Computer Communication & Collaboration (Vol. 7, Issue 1, 2019)
ISSN 2292-1028(Print) 2292-1036(Online)
DOIC:2292-1036-2019-02-001-97

Some three and four elements codes in Fibonacci languages[1]

Chunhua Cao (Correspondence author)
School of Mathematics and Statistics, Yunnan University
Kunming, Yunnan, 650091, China
E-mail: chhcao@ynu.edu.cn

Qin Liu, Ling Li, Yanling Li , Lijuan Zhu
School of Mathematics and Statistics, Yunnan University
Kunming, Yunnan, 650091, China

Abstract: In the theory of codes, it is known that the language $\{x, y\}$ is a code if and only if $xy \neq yx$. In 2004, Zheng-Zhu Li, Y.S. Tsai and Gian-Chi Yih provided a characterization for codes consisting of three words in Fibonacci languages. In this paper, we present some results of codes with three elements in the Fibonacci languages. Then we give a characterization on codes with four elements in Fibonacci languages.

Keywords: Three Element, Four Element, Code, Fibonacci Languages

2010 Mathematics Subject Classification: 68Q45; 68R15

1 Introduction and Preliminaries

Let X be a finite alphabet consisting of at least two letters and $|X|$ be the number of letters in X. Every finite string over X is called a word. The word that contains no letter is called the empty word, denoted by 1. The set of all words is denoted by X^*, which is a free monoid with concatenation. Let $X^+ = X^* \setminus \{1\}$. For any word $v \in X^*$, the length of v, denoted by $|v|$, is the number of letters occurring in v. In particular, $|1| = 0$. A word $x \in X^+$ is said to be primitive if it is not a proper power of another word. A word $u \in X^*$ is a prefix (or suffix) of a word $v \in X^*$ if there is a word $x \in X^*$ such that $ux = v$ (or $xu = v$). We write $u \leq_p v$ (or $u \leq_s v$) when

[1] This work is supported by a National Natural Science Foundation of China \# 11861071, an Applied Basic Research Program of Science and Technology Department of Yunnan Province of China \# 2014FB101, and an Educational Committee Major Natural Science Foundation of Yunnan Province of China \# ZD2015013. Corresponding author: Chunhua Cao, School of Mathematics and Statistics, Yunnan University, chhcao@ynu.edu.cn.

u is a prefix (or suffix) of v. By $u <_p v$, we mean $u \le_p v$ but $u \neq v$ and $v \neq 1$. A nonempty language $C \subseteq X^+$ is a code if $x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_n$, where $x_i, y_j \in C$ and $i = 1,2,\cdots,m$, $j = 1,2,\cdots,n$, implies that $m = n$ and $x_i = y_i$ for all $i = 1,\cdots,m$. (see [1])

We assume that $X = \{a,b\}$ and define some special codes in the Fibonacci languages. Let $w_1 = a$, $w_2 = b$ where $a,b \in X$. The two types of Fibonacci sequences are defined recursively as follows: (see [2])

$$w_1 = a, w_2 = b, w_3 = ab, \cdots, w_n = w_{n-2} w_{n-1}, w_{n+1} = w_{n-1} w_n, \cdots;$$

$$z_1 = a, z_2 = b, z_3 = ba, \cdots, z_n = z_{n-1} z_{n-2}, z_{n+1} = z_n z_{n-1}, \cdots.$$

Let $F_{a,b}^1$ be the set consisting of those words in the first sequence and $F_{a,b}^0$ be the set consisting of those words in the second sequence respectively.

That is

$$F_{a,b}^1 = \{w_1, w_2, w_3, w_4, w_5, \cdots\} = \{a, b, ab, bab, abbab, \cdots\}$$

and

$$F_{a,b}^0 = \{z_1, z_2, z_3, z_4, z_5, \cdots\} = \{a, b, ba, bab, babba, \cdots\}$$

And let

$$F_1^1 = \{w_n \in F_{a,b}^1 \mid n = 1,3,5,7,\cdots\} = \{w_1, w_3, w_5, w_7, \cdots\}$$
$$= \{a, ab, abbab, abbabbababbab, \cdots\};$$

$$F_2^1 = \{w_n \in F_{a,b}^1 \mid n = 2,4,6,8,\cdots\} = \{w_2, w_4, w_6, w_8, \cdots\}$$
$$= \{b, bab, bababbab, bababbababbabbababbab, \cdots\};$$

$$F_1^0 = \{z_n \in F_{a,b}^0 \mid n = 1,3,5,7,\cdots\} = \{z_1, z_3, z_5, z_7, \cdots\}$$
$$= \{a, ba, babba, babbababbabba, \cdots\};$$

$$F_2^0 = \{z_n \in F_{a,b}^0 \mid n = 2,4,6,8,\cdots\} = \{z_2, z_4, z_6, z_8, \cdots\}$$
$$= \{b, bab, babbabab, babbababbabbababbabab, \cdots\}.$$

Remarks. The Fibonacci words have following properties:

(1) No words in F_1^1 is a prefix of any words in F_2^1, vice versa.

(2) For all $w_i \in F_1^1$, $w_j \in F_2^1$, we have $a \le_p w_i$, $b \le_p w_j$.

(3) The Fibonacci languages F_1^1 and F_2^1 are codes.

(4) If $w_n \in F_{a,b}^1$ for $n \ge 3$, then $w_n = w_k w_{k+1} w_{k+3} w_{k+5} \cdots w_{n-1}$ for any $k < n$, $2 \mid n-k$.

(5) For Fibonacci words w_n, w_{n+2}, w_{n+3} in $F_{a,b}^1$, we have $w_{n+2} w_{n+2} = w_n w_{n+3}$.

(6) Let $\{w_r, w_{r+1}, w_t\} \subseteq F^1_{a,b}$, where $r+1<t$. Then $w_t \in \{w_r, w_{r+1}\}^+$.

The $F^0_{a,b}$ has same properties as above.

Definitions which are used in the paper but not stated here can be found in [1]. If $|X|=1$, we know any three-element languages is not a code. So in this paper, we always let $|X| \geq 2$.

2 Main result

In this section we characterize three-element and four-element codes which are Fibonacci words. In these two types of Fibonacci languages $F^1_{a,b}$ and $F^0_{a,b}$, the corresponding terms of Fibonacci words have the same length. We know that they are in fact a reverse pair. (see [1])

Lemma 2.1.[3] Let $\{w_r, w_s, w_t\} \subseteq F^1_{a,b}$, where $r<s<t$. Then $A=\{w_r, w_s, w_t\}$ is a code if and only if the following two conditions hold:

(1) $s \neq r+1$;

(2) $(s,t) \neq (r+2, r+3)$.

Lemma 2.2. [3] $(F^1_{a,b})^{(n)}$ is a prefix code for $n \geq 2$.

Lemma 2.3. [3] $(F^1_{a,b})^{(m)}$ is a prefix code for $m \geq 3$.

The Fibonacci languages $F^1_1(F^0_1)$ and $F^1_2(F^0_2)$ are codes, in addition, $(F^1_{a,b})^{(n)}$ is a biffix code for $n \geq 3$. A language containing three elements only in $F^1_1(F^0_1)$ or $F^1_2(F^0_2)$ or $(F^1_{a,b})^{(n)}$ is a code. Then we are thinking about a language which both contains words in $(F^1_{a,b})^{(n)}$ and $F^1_{a,b}$ is a code or not. And we also want to know the language which contains four Fibonacci words is a code or not.

Proposition 2.4. Let $A=\{w^n_r, w_s, w_t\}$ and $\{w_r, w_s, w_t\} \subseteq F^1_{a,b}$, where $r<s<t$ and $n \geq 2$. Then A is a code.

Proof. Suppose A is not a code , then there exists a word $z \in X^+$ with minimal length, which has two representations over A. That is, $z = x_1 x_2 \Lambda\ x_p = y_1 y_2 \Lambda\ y_q$, $x_i, y_j \in A$ with $x_1 \neq y_1$, $i=1,2,\Lambda\ ,p$ and $j=1,2,\Lambda\ q$. Since $x_1 <_p y_1$ or $y_1 <_p x_1$, we have $\{x_1, y_1\} \subseteq F^1_1$ or F^1_2. Without loss of generality, we may suppose $\{x_1, y_1\} \subseteq F^1_1$.

(1) If $x_1 = w^n_r, y_1 = w_s$, then $w^n_r x_2 \Lambda\ x_p = w_s y_2 \Lambda\ y_q$. By remark(4), we have $w_s = w_r w_{r+1} \Lambda$,

so $w_r w_r^{n-1} x_2 \Lambda\ x_p = w_r w_{r+1} \Lambda\ y_2 \Lambda\ y_q$, then $w_r^{n-1} x_2 \Lambda\ x_p = w_{r+1} \Lambda\ y_2 \Lambda\ y_q$. Since $a <_p w_r$ but $b <_p w_{r+1}$, which is a contradiction.

(2) If $x_1 = w_r^n, y_1 = w_t$, then $w_r^n x_2 \Lambda\ x_p = w_t y_2 \Lambda\ y_q$. Since $t - r \geq 2$, then $w_t = w_r w_{r+1} \Lambda$, we have a contradiction like (1).

(3) If $x_1 = w_s, y_1 = w_t$, then $w_s x_2 \Lambda\ x_p = w_t y_2 \Lambda\ y_q$. Since $w_t = w_s w_{s+1} \Lambda$, this implies that $x_2 \Lambda\ x_p = w_{s+1} \Lambda\ w_{t-1} y_2 \Lambda\ y_q$. If $x_2 \in F_1^1$, then $a <_p x_2$, which contradicts that $b <_p w_{s+1}$. If $x_2 \in F_2^1$, then $x_2 = w_r^n$ and $b <_p x_2$. Since $w_{s+1} = w_r w_{r+1} \Lambda$, so $w_r^n x_3 \Lambda\ x_p = w_r w_{r+1} \Lambda\ y_p$, we have a contradiction like (1).

Proposition 2.5. Let $A = \{w_r, w_s^n, w_t\}$ and $\{w_r, w_s, w_t\} \subseteq F_{a,b}^1$, where $r < s < t$, and $n \geq 2$. Then A is a code if and only if $(s,t) \neq (r+2, r+3)$.

Proof. $(\Rightarrow)$ Immediately.

$(\Leftarrow)$ Suppose the condition holds and A is not a code , then $x_1 x_2 \Lambda\ x_p = y_1 y_2 \Lambda\ y_q$ for some $x_1, x_2, \Lambda\ , x_p, y_1, y_2, \Lambda\ y_q \in A$ with $x_1 \neq y_1$. It's similar to proposition 2.4, we suppose that $\{x_1, y_1\} \subseteq F_1^1$.

(1) If $x_1 = w_r, y_1 = w_s^n$, then $w_r x_2 \Lambda\ x_p = w_s^n y_2 \Lambda\ y_q$. Since $w_s = w_r w_{r+1} \Lambda$, so $x_2 \Lambda\ x_p = w_{r+1} \Lambda\ w_s^{n-1} y_2 \Lambda\ y_q$. If $x_2 \in F_1^1$, then $a <_p x_2$, which contradicts that $b <_p w_{r+1}$. If $x_2 \in F_2^1$, then $x_2 = w_t$ and $b <_p w_t$, this implies that $w_t x_3 \Lambda\ x_p = w_{r+1} \Lambda\ w_s^{n-1} y_2 \Lambda\ y_q$.

(1-1) If $s - r \geq 4$, then $w_s = w_r w_{r+1} w_{r+3} \Lambda$ and $w_t = w_{r+1} w_{r+2} w_{r+4} \Lambda\ w_{t-1}$. Therefore we have $w_{r+1} w_{r+2} \Lambda\ x_3 \Lambda\ x_p = w_{r+1} w_{r+3} \Lambda\ w_s^{n-1} y_2 \Lambda\ y_q$. That is, $w_{r+2} \Lambda\ x_3 \Lambda\ x_p = w_{r+3} \Lambda\ y_2 \Lambda\ y_q$, since $a <_p w_{r+2}$ but $b <_p w_{r+3}$, which is a contradiction.

(1-2) If $s - r = 2$, then $w_s = w_r w_{r+1}$ and $w_t x_3 \Lambda\ x_p = w_{r+1} w_s^{n-1} y_2 \Lambda\ y_q$.

(1-2-1) When $n = 2$. If $t = r+3$, then $A = \{w_r, w_{r+2}^2, w_{r+3}\}$, this contradicts the condition. If $t \geq r+5$, then $w_t = w_{r+1} w_{r+2} w_{r+4} \Lambda$, and it must have $w_{r+1} w_{r+2} w_{r+4} \Lambda\ x_p = w_{r+1} w_{r+2} y_2 \Lambda\ y_q$. So $y_2 = w_r$ or $y_2 = w_s^2$. If $y_2 = w_r$, then $w_{r+1} w_{r+3} \Lambda\ x_3 \Lambda\ x_p = y_3 \Lambda\ y_q$. So $y_3 = w_t$ and we will have $w_{r+1} w_{r+3} \Lambda\ x_3 \Lambda\ x_p = w_{r+1} w_{r+2} \Lambda\ y_4 \Lambda\ y_q$, since $a <_p w_{r+2}$ but $b <_p w_{r+3}$, which is a contradiction. If $y_2 = w_s^2 = w_{r+2} w_{r+2}$, then $w_{r+2} w_{r+3} \Lambda\ w_{t-1} x_3 \Lambda\ x_p = w_{r+2} w_{r+2} y_3 \Lambda\ y_q$, since $a <_p w_{r+2}$ but $b <_p w_{r+3}$, which is a contradiction.

(1-2-2) When $n = 3$. If $t = r+3, w_t = w_{r+1} w_{r+2}$, then $A = \{w_r, w_{r+2}^3, w_{r+3}\}$, this contradicts the condition. If $t = r+5$, then $w_t = w_{r+1} w_{r+2} w_{r+4}$, so $w_{r+1} w_{r+2} w_{r+4} x_3 \Lambda\ x_p = w_{r+1} w_{r+2}^2 y_2 \Lambda\ y_q$. Hence $y_2 = w_t = w_{r+3} w_{r+4}$, then $x_3 \Lambda\ x_p = w_{r+4} y_3 \Lambda\ y_q$. So $x_3 = w_r$ or $x_3 = w_s^3$. If $x_3 = w_r$, then $w_r x_4 \Lambda\ x_p = w_r w_{r+1} w_{r+3} y_3 \Lambda\ y_q$, we have $x_4 = w_t$. It follows that $w_{r+1} w_{r+2} w_{r+4} x_5 \Lambda\ x_p = w_{r+1} w_{r+3} y_3 \Lambda\ y_q$, since $a <_p w_{r+2}$ but $b <_p w_{r+3}$, which is a contradiction. If $x_3 = w_s^3$, then $w_r w_{r+1} w_{r+2}^2 x_4 \Lambda\ x_p = w_r w_{r+1} w_{r+3} y_3 \Lambda\ y_q$, a contradiction similar to $x_3 = w_r$. If $t \geq r+7$, then $w_t = w_{r+1} w_{r+2} w_{r+4} w_{r+6} \Lambda$ and $w_{r+1} w_{r+2} w_{r+4} w_{r+6} \Lambda\ x_3 \Lambda\ x_p = w_{r+1} w_{r+2}^2 y_2 \Lambda\ y_q$. So $y_2 = w_t$. We have $w_{r+3} w_{r+6} \Lambda\ x_3 \Lambda\ x_p = w_{r+3} w_{r+4} w_{r+6} \Lambda\ y_3 \Lambda\ y_q = w_{r+3} w_{r+4} w_{r+5} \Lambda\ x_3 \Lambda\ x_p$, since $a <_p w_{r+6}$ but $b <_p w_{r+5}$, which is a contradiction.

(1-2-3) When $n \geq 4$. If $t = r+3$, then $w_t = w_{r+1} w_{r+2}$, $A = \{w_r, w_{r+2}^n, w_{r+3}\}$, this contradicts the condition. If $t \geq r+5$, then $w_t = w_{r+1} w_{r+2} w_{r+4} \Lambda$, and $w_{r+1} w_{r+2} w_{r+4} \Lambda\ x_p = w_{r+1} w_{r+2}^{n-1} \Lambda\ y_q$, that is $w_{r+3} \Lambda\ x_p = w_{r+2}^{n-3} \Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+2}$ and $b <_p w_{r+3}$.

(2) If $x_1 = w_r, y_1 = w_t$, then $w_r x_2 \Lambda\ x_p = w_t y_2 \Lambda\ y_q$. Since $t - r \geq 2$, so $w_t = w_r w_{r+1} \Lambda$, we must have $w_r x_2 \Lambda\ x_p = w_r w_{r+1} \Lambda\ w_{t-1} y_2 \Lambda\ y_q$. If $x_2 \in F_1^1$, then $a <_p x_2$, which contradicts $b <_p w_{r+1}$. If $x_2 \in F_2^1$, then $b <_p x_2$ and $x_2 = w_s^n$, so $w_s^n x_3 \Lambda\ x_p = w_{r+1} \Lambda\ w_{t-1} y_2 \Lambda\ y_q$.

(2-1) If $t - r = 2$, then $s = r+1$, and $x_2 \Lambda\ x_p = w_{r+1} y_2 \Lambda\ y_q$. So $x_2 = w_s^n$, then we obtain $y_2 = w_s^n$, continuing this process, we know that $x_i = y_i = w_s^n$ for all $i \geq 2$, which indicates the length of the word $|z|$ is infinite, which is a contradiction.

(2-2) If $t - r = 4$, then $w_t = w_r w_{r+1} w_{r+3}$.

(2-2-1) When $n = 2$. If $s = r+1$, then $w_{r+1}^2 x_3 \Lambda\ x_p = w_{r+1} w_{r+1} w_{r+2} y_2 \Lambda\ y_q$. So $x_3 = w_r$ or $x_3 = w_t$. If $x_3 = w_r$, then $x_4 \Lambda\ x_p = w_{r+1} y_2 \Lambda\ y_q$. We must have $x_4 = w_s^2$. Therefore $w_{r+1}^2 x_5 \Lambda\ x_p = w_{r+1} y_2 \Lambda\ y_q$, so $y_2 = w_s^2$. Continuing this process, we obtain that $x_i = w_s^2, i \geq 4$ and $y_j = w_s^2$, $j \geq 2$, which contradicts the finite length of $|z|$. If $x_3 = w_t$, then we have $w_r w_{r+1} w_{r+3} x_4 \Lambda\ x_p = w_{r+2} y_2 \Lambda\ y_q$. So $y_2 = w_s^n$, then $w_{r+1} w_{r+2} x_4 \Lambda\ x_p = w_{r+1}^2 y_3 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+2}$ and $b <_p w_{r+1}$. If $s = r+3$, then $w_s = w_{r+1} w_{r+2}$, then $w_{r+1} w_{r+2} w_s x_3 \Lambda\ x_p = w_{r+1} w_{r+3} y_2 \Lambda\ y_q$. Since $a <_p w_{r+2}$ but $b <_p w_{r+3}$, which is a contradiction.

(2-2-2) When $n \geq 3$. If $s = r+1$ or $s = r+3$, then $w_{r+1}^n x_3 \Lambda\ x_p = w_{r+1} w_{r+1} w_{r+2} y_2 \Lambda\ y_q$ or $w_{r+2} w_s^{n-1} x_3 \Lambda\ x_p = w_{r+1} w_{r+2} y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+2}$ and $b <_p w_{r+1}$.

(2-3) If $t - r \geq 6$, then $w_t = w_r w_{r+1} w_{r+3} w_{r+5} \Lambda\ w_{t-1}$.

(2-3-1) If $n = 2$, then $w_r x_2 \Lambda\ x_p = w_r w_{r+1} w_{r+3} w_{r+5} \Lambda\ w_{t-1} y_2 \Lambda\ y_q$, so $x_2 = w_s^2$. If $s = r+1$, then $w_{r+1} w_{r+1} x_3 \Lambda\ x_p = w_{r+1} w_{r+1} w_{r+2} w_{r+5} \Lambda\ y_2 \Lambda\ y_q$, so $x_3 = w_r$ or $x_3 = w_t$. If $x_3 = w_t$, then $w_r w_{r+1} w_{r+3} w_{r+5} \Lambda\ x_4 \Lambda\ x_p = w_r w_{r+1} w_{r+5} \Lambda\ y_2 \Lambda\ y_q = w_r w_{r+1} w_{r+3} w_{r+4} \Lambda\ y_q$, since $a <_p w_{r+4}$ but $b <_p w_{r+5}$, which is a contradiction. If $x_3 = w_r$, then $x_4 \Lambda\ x_p = w_{r+1} w_{r+5} \Lambda\ y_2 \Lambda\ y_q$, so $x_4 = w_s^2$

and $w_s^2 x_5 \Lambda\ x_p = w_{r+1}w_{r+1}w_{r+2}w_{r+4}\ \Lambda\ y_2 \Lambda\ y_q$. So $x_5 = w_r$ or $x_5 = w_t$. If $x_5 = w_r$, then we obtain that $w_r x_6 \Lambda\ x_p = w_r w_{r+1} w_{r+4} \Lambda\ y_2 \Lambda\ y_q$, then $x_6 = w_s^2 = w_{r+1}^2$, it implies that $w_{r+1}^2 x_7 \Lambda\ x_p$ $w_{r+1}w_{r+2}\Lambda\ y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+2}$ and $b <_p w_{r+1}$. If $x_5 = w_t$, then $w_r w_{r+1} w_{r+3} w_{r+5} \Lambda\ x_6 \Lambda\ x_p = w_{r+2}w_{r+4}\Lambda\ y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+4}$ and $b <_p w_{r+3}$. If $s \geq r+3$, then $w_s = w_{r+1}w_{r+2}\Lambda$ and $w_{r+1}w_{r+2}\Lambda\ w_s x_3 \Lambda\ x_p =$ $w_{r+1}w_{r+3}\Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+2}$ and $b <_p w_{r+3}$.

(2-3-2) If $n \geq 3$ and $s = r+1$, then $w_{r+1}^n x_3 \Lambda\ x_p = w_{r+1}w_{r+3}w_{r+5}\Lambda\ y_q = w_{r+1}w_{r+1}w_{r+2}\Lambda\ y_q$, which contradicts the fact that $a <_p w_{r+2}$ and $b <_p w_{r+1}$. If $s \geq r+3$, then $w_s = w_{r+1}w_{r+2}\Lambda$, then $w_{r+1}w_{r+2}\Lambda\ w_s^{n-1}x_3 \Lambda\ x_p = w_{r+1}w_{r+3}w_{r+5}\Lambda\ y_2 \Lambda\ y_q$, since $a <_p w_{r+2}$ but $b <_p w_{r+3}$, which is a contradiction.

(3) If $x_1 = w_s^n, y_1 = w_t$, then we have $w_s^n x_2 \Lambda\ x_p = w_t y_2 \Lambda\ y_q$. For $w_t = w_s w_{s+1} \Lambda\ w_{t-1}$, then $w_s^n x_2 \Lambda\ x_p = w_s w_{s+1} \Lambda\ w_{t\text{-}1} y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_s$ and $b <_p w_{s+1}$.

Corollary 2.6. Let $A = \{w_r, w_s, w_t^n\}$ and $\{w_r, w_s, w_t\} \subseteq F_{a,b}^1$, where $r < s < t$ and $n \geq 2$. Then A is a code if and only if $s \neq r+1$.

Proposition 2.7. Let $A = \{w_r^n, w_s^n, w_t\}$ and $\{w_r, w_s, w_t\} \subseteq F_{a,b}^1$, where $r < s < t$ and $n \geq 2$. Then A is a code .

Proof. The process is the same as proof of proposition 2.4, we suppose that $\{x_1, y_1\} \subseteq F_1^1$.

(1) If $x_1 = w_r^n$, $y_1 = w_s^n$, then $w_r^n x_2 \Lambda\ x_p = w_s^n y_2 \Lambda\ y_q$. Since $w_s = w_r w_{r+1} \Lambda\ w_{s-1}$, we have $w_r^n x_2 \Lambda\ x_p = w_r w_{r+1} \Lambda\ w_s^{n-1} y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_r$ and $b <_p w_{r+1}$.

(2) If $x_1 = w_r^n$, $y_1 = w_t$, then $w_r^n x_2 \Lambda\ x_p = w_t y_2 \Lambda\ y_q$. Since $t - r \geq 2$, so $w_t = w_r w_{r+1} \Lambda\ w_{t-1}$, hence $w_r^n x_2 \Lambda\ x_p = w_r w_{r+1} \Lambda\ y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_r$ and $b <_p w_{r+1}$.

(3) If $x_1 = w_s^n$, $y_1 = w_t$, then $w_s^n x_2 \Lambda\ x_p = w_t y_2 \Lambda\ y_q$. Since $w_t = w_s w_{s+1} \Lambda\ w_{t-1}$, so we have $w_s^n x_2 \Lambda\ x_p = w_s w_{s+1} \Lambda\ y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_s$ and $b <_p w_{s+1}$.

Corollary 2.8. Let $A = \{w_r^n, w_s, w_t^n\}$ and $\{w_r, w_s, w_t\} \subseteq F_{a,b}^1$, where $r < s < t$ and $n \geq 2$. Then A is a code .

Corollary 2.9. Let $A = \{w_r, w_s^n, w_t^n\}$ and $\{w_r, w_s, w_t\} \subseteq F_{a,b}^1$, where $r < s < t$ and $n \geq 2$. Then A is a code .

Proposition 2.10. Let $A=\{w_r, w_s, w_t, w_k\}\subseteq F^1_{a,b}$, where $r<s<t<k$. Then A is a code if and only if the following four conditions are true:

(1) $s\neq r+1$;

(2) $t\neq s+1$;

(3) $(s,t)\neq(r+2, r+3)$;

(4) $(t,k)\neq(s+2, s+3)$.

Proof. ($\Rightarrow$) If $s=r+1$, then $w_t, w_k\in\{w_r, w_{r+1}\}^+$. If $t=s+1$, then $w_k\in\{w_s, w_{s+1}\}^+$. If $(s,t)=(r+2, r+3)$, then $w^2_{r+2}=w_r w_{r+3}$. If $(t,k)=(s+2, s+3)$, then $w^2_{s+2}=w_s w_{s+3}$. All contradicts that A is a code.

($\Leftarrow$) If $A\subseteq F^1_1$ or $A\subseteq F^1_2$, then A is a code. Let $A\not\subset F^1_1$ and $A\not\subset F^1_2$, we show that A is a code. Suppose the conditions hold and A is not a code, then there exists a word w with minimal length such that $w=x_1x_2\Lambda\ x_p=y_1y_2\Lambda\ y_q$, where $x_1,x_2,\Lambda\ ,x_p,y_1,y_2,\Lambda\ ,y_q\in A$, $x_1\neq y_1$. Since $x_1<_p y_1$ or $y_1<_p x_1$, we have $\{x_1,y_1\}\subseteq F^1_1$ or $\{x_1,y_1\}\subseteq F^1_2$. Without loss of generality , we assume that $\{x_1,y_1\}\subseteq F^1_1$ and $u_1,v_1\in A\backslash\{x_1,y_1\}$. There are two cases: (1) $u_1,v_1\in F^1_2$; (2) $u_1\in F^1_1$ and $v_1\in F^1_2$. Let $x_1=w_i$, $y_1=w_j$, $u_1=w_m$, $v_1=w_n$ and $i<j$, $2\,|\,j-i$.

(1) If $\{x_1,y_1\}\subseteq F^1_1$ and $u_1,v_1\in F^1_2$, then $2\,|\,n-m$. Let $m<n$.

(1-1) If $j-i\geq 4$, by remark (4), we have $w_j=w_iw_{i+1}w_{i+3}\Lambda$, then $x_2\Lambda\ x_p=w_{i+1}w_{i+3}\Lambda\ y_q$. So $x_2=w_m$ or $x_2=w_n$.

(1-1-1) If $x_2=w_m$, then $w_mx_3\Lambda\ x_p=w_{i+1}w_{i+3}\Lambda\ y_2\Lambda\ y_q$.

(1-1-1-1) If $m=i+1$ or $m=i-1$, then $A=\{w_i,w_{i+1},w_j,w_n\}$ or $A=\{w_{i-1},w_i,w_j,w_n\}$, which contradicts condition (1).

(1-1-1-2) If $m\leq i-3$, then $w_{i+1}=w_mw_{m+1}w_{m+3}\Lambda$, we have $w_mw_{m+1}w_{m+3}\leq_p w_{i+1}$. Therefore $w_mx_3\Lambda\ x_p=w_mw_{m+1}w_{m+3}\Lambda\ w_{i+3}\Lambda\ y_2\Lambda\ y_q$, then $x_3=w_i$ or $x_3=w_j$. Since $w_i,w_j\in F^1_1$, we have $w_{m+1}w_{m+2}\leq_p w_i$ and $w_{m+1}w_{m+2}\leq_p w_j$, then $w_{m+1}w_{m+2}\Lambda\ x_4\Lambda\ x_p=w_{m+1}w_{m+3}\Lambda\ y_q$. Since $b<_p w_{m+2}$ but $a<_p w_{m+3}$, which is a contradiction.

(1-1-1-3) If $m\geq i+3$, then $x_2=w_m=w_{i+1}w_{i+2}\Lambda$. So $w_{i+1}w_{i+2}\Lambda\ x_3\Lambda\ x_p=w_{i+1}w_{i+3}\Lambda\ y_q$, which contradicts the fact that $a<_p w_{i+2}$ and $b<_p w_{i+3}$.

(1-1-2) If $x_2=w_n$, then $w_nx_3\Lambda\ x_p=w_{i+1}w_{i+3}\Lambda\ y_2\Lambda\ y_q$.

(1-1-2-1) If $n=i+1$ or $n=i-1$, then $A=\{w_m,w_i,w_{i+1},w_j\}$ or $A=\{w_m,w_{i-1},w_i,w_j\}$, which contradicts condition (2).

(1-1-2-2) If $n \le i-3$, then $w_{i+1} = w_n w_{n+1} w_{n+3} \Lambda$. So $x_3 \Lambda\ x_p = w_{n+1} w_{n+3} \Lambda\ w_{i+3} \Lambda\ y_2 \Lambda\ y_q$. It must have $x_3 = w_i$ or $x_3 = w_j$. Since $w_i, w_j \in F_1^1$, we have $w_{n+1} w_{n+2} \le_p w_i$ and $w_{n+1} w_{n+2} \le_p w_j$, then $w_{n+1} w_{n+2} \Lambda\ x_4 \Lambda\ x_p = w_{n+1} w_{n+3} \Lambda\ w_{i+3} \Lambda\ y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{n+3}$ and $b <_p w_{n+2}$.

(1-1-2-3) If $n \ge i+3$, then $w_n = w_{i+1} w_{i+2} \Lambda$. Hence $w_{i+1} w_{i+2} \Lambda\ x_p = w_{i+1} w_{i+3} \Lambda\ y_q$, which contradicts $a <_p w_{i+2}$ and $b <_p w_{i+3}$.

(1-2) If $j-i=2$, then $w_j = w_{i+2} = w_i w_{i+1}$ and we have $x_2 \Lambda\ x_p = w_{i+1} y_2 \Lambda\ y_q$. Therefore $x_2 = w_m$ or $x_2 = w_n$.

(1-2-1) If $x_2 = w_m$, then $w_m x_3 \Lambda\ x_p = w_{i+1} y_2 \Lambda\ y_q$.

(1-2-1-1) If $m = i+1$ or $m = i-1$, then $A = \{w_i, w_{i+1}, w_{i+2}, w_n\}$ or $A = \{w_{i-1}, w_i, w_{i+2}, w_n\}$, which contradicts condition (1).

(1-2-1-2) If $m \le i-3$, then $w_{i+1} = w_m w_{m+1} w_{m+3} \Lambda$, we have $w_m w_{m+1} w_{m+3} \le_p w_{i+1}$. Therefore $w_m x_3 \Lambda\ x_p = w_m w_{m+1} w_{m+3} \Lambda\ w_i y_2 \Lambda\ y_q$, then $x_3 = w_i$ or $x_3 = w_j$. Since $w_i, w_j \in F_1^1$, we have $w_{m+1} w_{m+2} \le_p w_i$ and $w_{m+1} w_{m+2} \le_p w_j$, then $w_{m+1} w_{m+2} \Lambda\ x_4 \Lambda\ x_p = w_{m+1} w_{m+3} \Lambda\ y_2 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{m+3}$ and $b <_p w_{m+2}$.

(1-2-1-3) If $m = i+3$, then $A = \{w_i, w_{i+2}, w_{i+3}, w_n\}$, which contradicts condition (3).

(1-2-1-4) If $m = i+5$, then $w_m = w_{i+1} w_{i+2} w_{i+4}$. We have $w_{i+1} w_{i+2} w_{i+4} x_3 \Lambda\ x_p = w_{i+1} y_2 \Lambda\ y_q$. Hence $y_2 = w_i$ or $y_2 = w_j$. If $y_2 = w_i$, then $w_{i+1} w_{i+4} x_3 \Lambda\ x_p = y_3 \Lambda\ y_q$. Therefore $y_3 = w_m$ or $y_3 = w_n$. Since $w_m, w_n \in F_2^1$, we have $w_{i+1} w_{i+2} w_{i+4} \le_p w_m$ and $w_{i+1} w_{i+2} w_{i+4} \le_p w_n$, then $w_{i+1} w_{i+4} x_3 \Lambda\ x_p = w_{i+1} w_{i+2} w_{i+4} \Lambda\ y_4 \Lambda\ y_q = w_{i+1} w_{i+2} w_{i+3} x_3 \Lambda\ x_p$, since $a <_p w_{i+4}$ but $b <_p w_{i+3}$, which is a contradiction. If $y_2 = w_j$, then $w_{i+4} x_3 \Lambda\ x_p = y_3 \Lambda\ y_q$. So $y_3 = w_i$ or $y_3 = w_j$. If $y_3 = w_i$, then $w_{i+1} w_{i+3} x_3 \Lambda\ x_p = y_4 \Lambda\ y_q$. So $y_4 = w_m$ or $y_4 = w_n$. Since $w_{i+1} w_{i+2} \le_p w_m$ and $w_{i+1} w_{i+2} \le_p w_n$, we have $w_{i+1} w_{i+3} x_3 \Lambda\ x_p = w_{i+1} w_{i+2} \Lambda\ y_q$, since $a <_p w_{i+2}$ but $b <_p w_{i+3}$, which is a contradiction. If $y_3 = w_j$, then $w_{i+3} x_3 \Lambda\ x_p = y_4 \Lambda\ y_q$. So $y_4 = w_m$ or $y_4 = w_n$. If $y_4 = w_m$, then $x_3 \Lambda\ x_p = w_{i+4} y_5 \Lambda\ y_q$. We obtain that $x_3 = w_i$ or $x_3 = w_j$. If $x_3 = w_i$, then we have $x_4 \Lambda\ x_p = w_{i+1} w_{i+3} y_5 \Lambda\ y_q$. It follows that $x_4 = w_m$ or $x_4 = w_n$, then we have same contradictions as above. If $x_3 = w_j$, similar to the above, we have $x_4 = w_m$ or $x_4 = w_n$ and then $y_5 = w_i$ or $y_5 = w_j$. Continuing this process, we have $x_s = y_s = w_j$, $x_{s+1} = y_{s+1} = w_m$ for $s \ge 3$ and s is an odd number. This implies that the length of the word $|w|$ is infinite, which is a contradiction. And $y_4 = w_n$ has the same contradiction with $y_4 = w_m$.

(1-2-1-5) If $m \ge i+7$, then $w_m = w_{i+1} w_{i+2} w_{i+4} w_{i+6} \Lambda$. So $w_{i+2} w_{i+4} w_{i+6} \Lambda\ x_3 \Lambda\ x_p = y_2 \Lambda\ y_q$. We have $y_2 = w_i$ or $y_2 = w_j$. If $y_2 = w_i$, then $w_{i+1} w_{i+4} w_{i+6} \Lambda\ x_3 \Lambda\ x_p = y_3 \Lambda\ y_q$, so we have $y_3 = w_m$ or $y_3 = w_n$. Since $w_{i+1} w_{i+2} w_{i+4} \le_p w_m$ and $w_{i+1} w_{i+2} w_{i+4} \le_p w_n$. Therefore we have

$w_{i+1}w_{i+4}w_{i+6}\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}\Lambda\ y_4\Lambda\ y_q$, which is a contradiction. If $y_2 = w_j$, then we have $w_{i+4}w_{i+6}\Lambda\ x_3\Lambda\ x_p = y_3\Lambda\ y_q$, so $y_3 = w_i$ or $y_3 = w_j$. If $y_3 = w_i$, then $w_{i+1}w_{i+3}w_{i+6}\Lambda\ x_p = y_4\Lambda\ y_q$, so we have $y_4 = w_m$ or $y_4 = w_n$. Since $w_{i+1}w_{i+2} \leq_p w_m$ and $w_{i+1}w_{i+2} \leq_p w_n$, we have $w_{i+1}w_{i+3}w_{i+6}\Lambda\ x_p = w_{i+1}w_{i+2}\Lambda\ y_5\Lambda\ y_q$, a contradiction. If $y_3 = w_j$, then $w_{i+3}w_{i+6}\Lambda\ x_p = y_4\Lambda\ y_q$, so $y_4 = w_m$ or $y_4 = w_n$. We have $w_{i+3}w_{i+6}\Lambda\ x_p = w_{i+3}w_{i+4}w_{i+6}\Lambda\ y_5\Lambda\ y_q$, which is a contradiction.

(1-2-2) If $x_2 = w_n$, then $w_n x_3\Lambda\ x_p = w_{i+1}y_2\Lambda\ y_q$.

(1-2-2-1) If $n = i+1$ or $n = i-1$, then $A = \{w_m, w_i, w_{i+1}, w_{i+2}\}$ or $A = \{w_m, w_{i-1}, w_i, w_{i+2}\}$, which contradicts condition (2).

(1-2-2-2) If $n \leq i-3$, the proof is the same as (1-1-2-2).

(1-2-2-3) If $n = i+3$, then $A = \{w_i, w_m, w_{i+2}, w_{i+3}\}$. If $m = i+1$, then $A = \{w_i, w_{1+1}, w_{i+2}, w_{i+3}\}$. If $m = i-1$, then $A = \{w_i, w_m, w_{i+2}, w_{i+3}\}$, both contradicts condition (1). If $m \leq i-3$, then $A = \{w_m, w_i, w_{i+2}, w_{i+3}\}$, which contradicts condition (4).

(1-2-2-4) If $n = i+5$, we have $w_n = w_{i+1}w_{i+2}w_{i+4}$ and $A = \{w_i, w_m, w_{i+2}, w_{i+5}\}$. If $m = i+3$, then $A = \{w_i, w_{i+2}, w_{i+3}, w_{i+5}\}$, a contradiction with condition (3). If $m = i+1$ or $m = i-1$, then $A = \{w_i, w_{i+1}, w_{i+2}, w_{i+5}\}$ or $A = \{w_{i-1}, w_i, w_{i+2}, w_{i+5}\}$, which is a contradiction with condition (1). If $m \leq i-3$, we have $w_{i+2}w_{i+4}x_3\Lambda\ x_p = y_2\Lambda\ y_q$. So $y_2 = w_i$ or $y_2 = w_j$. If $y_2 = w_i$, then $w_{i+1}w_{i+4}x_3\Lambda\ x_p = y_3\Lambda\ y_q$, so $y_3 = w_m$ or $y_3 = w_n$. If $y_3 = w_n = w_{i+1}w_{i+2}w_{i+4}$, then $w_{i+1}w_{i+4}x_3\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}y_4\Lambda\ y_q = w_{i+1}w_{i+2}w_{i+3}x_3\Lambda\ x_p$ which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_3 = w_m$, then $w_{i+1} = w_m w_{m+1}w_{m+3}\Lambda$. We have $w_{m+1}w_{m+3}\Lambda\ x_p = y_4\Lambda\ y_q$. Hence $y_4 = w_i$ or $y_4 = w_j$. Since $w_{m+1}w_{m+2} \leq_p w_i$ and $w_{m+1}w_{m+2} \leq_p w_j$, we have $w_{m+1}w_{m+3}\Lambda\ x_p = w_{m+1}w_{m+2}\Lambda\ y_5\Lambda\ y_q$ which contradicts that $b <_p w_{m+2}$ and $a <_p w_{m+3}$. If $y_2 = w_j$, then $w_{i+4}x_3\Lambda\ x_p = y_3\Lambda\ y_q$. So $y_3 = w_i$ or $y_3 = w_j$. If $y_3 = w_i$, then $y_4\Lambda\ y_q = w_{i+1}w_{i+3}x_3\Lambda\ x_p$, so we have $y_4 = w_m$ or $y_4 = w_n$. If $y_4 = w_n = w_{i+1}w_{i+2}w_{i+4}$, we must have $w_{i+1}w_{i+2}w_{i+4}y_5\Lambda\ y_q = w_{i+1}w_{i+3}x_3\Lambda\ x_p$, since $b <_p w_{i+3}$ but $a <_p w_{i+2}$, which is a contradiction. If $y_4 = w_m$, then $w_{i+1} = w_m w_{m+1}w_{m+3}\Lambda$, we have $w_{m+1}w_{m+3}\Lambda\ w_{i+3}x_3\Lambda\ x_p = y_5\Lambda\ y_q$. Therefore $y_5 = w_i$ or $y_5 = w_j$. Since $w_{m+1}w_{m+2} \leq_p w_i$ and $w_{m+1}w_{m+2} \leq_p w_j$, we have contradictions like above. If $y_3 = w_j$, then $w_{i+3}x_3\Lambda\ x_p = y_4\Lambda\ y_q$. So $y_4 = w_m$ or $y_4 = w_n$. If $y_4 = w_m$, then we have $w_{i+3} = w_m w_{m+1}w_{m+3}\Lambda$ and $w_{m+1}w_{m+3}\Lambda\ x_3\Lambda\ x_p = y_5\Lambda\ y_q$. Therefore $y_5 = w_i$ or $y_5 = w_j$. Since $w_{m+1}w_{m+2} \leq_p w_i$ and $w_{m+1}w_{m+2} \leq_p w_j$, we have contradictions like above. If $y_4 = w_n = w_{i+3}w_{i+4}$, then $x_3\Lambda\ x_p = w_{i+4}y_5\Lambda\ y_q$. So $x_3 = w_i$ or $x_3 = w_j$. If $x_3 = w_i$, then $w_{i+1}w_{i+3}y_5\Lambda\ y_q = x_4\Lambda\ x_p$. So $x_4 = w_m$ or $x_4 = w_n$. If $x_4 = w_n = w_{i+1}w_{i+2}w_{i+4}$, then we have $w_{i+1}w_{i+3}y_5\Lambda\ y_q = w_{i+1}w_{i+2}w_{i+4}x_5\Lambda\ x_p$, which contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$.

If $x_4 = w_m$, then $w_{i+1} = w_m w_{m+1} w_{m+3} \Lambda$ and $x_5 \Lambda\ x_p = w_{m+1} w_{m+3} \Lambda\ w_{i+3} y_5 \Lambda\ y_q$. So $x_5 = w_i$ or $x_5 = w_j$. Since $w_{m+1} w_{m+2} \leq_p w_i$ and $w_{m+1} w_{m+2} \leq_p w_j$, we have contradictions like above. If $x_3 = w_j$, then $x_4 \Lambda\ x_p = w_{i+3} y_5 \Lambda\ y_q$. So $x_4 = w_m$ or $x_4 = w_n$. If $x_4 = w_m$, then $w_{i+3} = w_m w_{m+1} w_{m+3} \Lambda$. so we have $x_5 = w_i$ or $x_5 = w_j$. Since $w_{m+1} w_{m+2} \leq_p w_i$ and $w_{m+1} w_{m+2} \leq_p w_j$, we have contradictions as above. If $x_4 = w_n$, then $w_{i+4} x_5 \Lambda\ x_p = y_5 \Lambda\ y_q$. Therefore $y_5 = w_i$ or $y_5 = w_j$. Continuing this process, we know $x_{s+1} = y_{s+1} = w_n$ and $x_s = y_s = w_j$ for $s \geq 3$ and s is an odd number. This implies the length of $|w|$ is infinite, which is a contradiction.

(1-2-2-5) If $n \geq i+7$, then $w_n = w_{i+1} w_{i+2} w_{i+4} \Lambda$. We have $w_{i+2} w_{i+4} \Lambda\ x_3 \Lambda\ x_p = y_2 \Lambda\ y_q$. So $y_2 = w_i$ or $y_2 = w_j$. If $y_2 = w_i$, then $w_{i+1} w_{i+4} \Lambda\ x_3 \Lambda\ x_p = y_3 \Lambda\ y_q$, so $y_3 = w_m$ or $y_3 = w_n$. If $y_3 = w_n = w_{i+1} w_{i+2} w_{i+4}$, then $w_{i+1} w_{i+4} \Lambda\ x_3 \Lambda\ x_p = w_{i+1} w_{i+2} w_{i+4} y_4 \Lambda\ y_q = w_{i+1} w_{i+2} w_{i+3} \Lambda\ x_p$, which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_3 = w_m$, we compare the size of m and $i+1$. If $m \geq i+5$, then $w_m = w_{i+1} w_{i+2} w_{i+4} \Lambda$ and we have a same contradiction as $y_3 = w_n$. If $m = i+3$, then $A = \{w_i, w_{i+2}, w_{i+3}, w_n\}$, which is a contradiction with condition (3). If $m = i+1$ or $m = i-1$, then $A = \{w_i, w_{i+1}, w_{i+2}, w_n\}$ or $A = \{w_{i-1}, w_i, w_{i+2}, w_n\}$, which are contradictions with condition (3) and (1). If $m \leq i-3$, then $w_{i+1} = w_m w_{m+1} w_{m+3} \Lambda$. So we have $w_{m+1} w_{m+3} \Lambda\ w_{i+4} \Lambda\ x_3 \Lambda\ x_p = y_4 \Lambda\ y_q$. Hence $y_4 = w_i$ or $y_4 = w_j$. Since $w_{m+1} w_{m+2}$ is a prefix of w_i and w_j, then we have $w_{m+1} w_{m+3} \Lambda\ w_{i+4} \Lambda\ x_3 \Lambda\ x_p = w_{m+1} w_{m+2} \Lambda\ y_5 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{m+3}$ and $b <_p w_{m+2}$. If $y_2 = w_j$, then $w_{i+4} \Lambda\ x_3 \Lambda\ x_p = y_3 \Lambda\ y_q$. So $y_3 = w_i$ or $y_3 = w_j$. If $y_3 = w_i$, then $y_4 \Lambda\ y_q = w_{i+1} w_{i+3} \Lambda\ x_3 \Lambda\ x_p$, so we have $y_4 = w_m$ or $y_4 = w_n$. If $y_4 = w_n = w_{i+1} w_{i+2} w_{i+4} \Lambda$, then $w_{i+1} w_{i+2} w_{i+4} \Lambda\ y_5 \Lambda\ y_q = w_{i+1} w_{i+3} \Lambda\ x_3 \Lambda\ x_p$, which contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$. If $y_4 = w_m$, then we have $w_{i+1} = w_m w_{m+1} w_{m+3} \Lambda$, so $w_{m+1} w_{m+3} \Lambda\ w_{i+3} \Lambda\ x_3 \Lambda\ x_p = y_5 \Lambda\ y_q$. Hence $y_5 = w_i$ or $y_5 = w_j$. Since $w_{m+1} w_{m+2} \leq_p w_i$ and $w_{m+1} w_{m+2} \leq_p w_j$, we have a contradiction like $y_4 = w_i$. If $y_3 = w_j$, then $w_{i+3} w_{i+6} \Lambda\ x_3 \Lambda\ x_p = y_4 \Lambda\ y_q$. So $y_4 = w_m$ or $y_4 = w_n$. If $y_4 = w_n = w_{i+3} w_{i+4} w_{i+6} \Lambda$, then $w_{i+3} w_{i+4} w_{i+5} \Lambda\ x_3 \Lambda\ x_p = w_{i+3} w_{i+4} w_{i+6} y_5 \Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+6}$ and $b <_p w_{i+5}$. If $y_4 = w_m$, then $w_{m+1} w_{m+3} \Lambda\ x_3 \Lambda\ x_p = y_5 \Lambda\ y_q$. Hence $y_5 = w_i$ or $y_5 = w_j$. Since $w_{m+1} w_{m+2} \leq_p w_i$ and $w_{m+1} w_{m+2} \leq_p w_j$, we have a contradiction like $y_4 = w_i$.

(2) If $x_1, y_1, u_1 \in F_1^1$ and $v_1 \in F_2^1$, then $2 | j-i$ and $2 | j-m$.

(2-1) If $j-i \geq 4$, then $w_j = w_i w_{i+1} w_{i+3} \mathrm{L}\ w_{j-1}$, then $x_2 \Lambda\ x_p = w_{i+1} w_{i+3} \Lambda\ y_q$. So $x_2 = w_n$.

(2-1-1) If $n = i+1$ or $n = i-1$, then $A = \{w_i, w_{i+1}, w_j, w_m\}$ or $A = \{w_{i-1}, w_i, w_j, w_m\}$, which contradicts condition (1).

(2-1-2) If $n \leq i-3$, then $w_{i+1} = w_n w_{n+1} w_{n+3} \mathrm{L}\ w_i$, we have $x_3 \Lambda\ x_p = w_{n+1} w_{n+3} \Lambda\ y_q$. It follows that $x_3 = w_i$ or $x_3 = w_j$ or $x_3 = w_m$. Since $w_i, w_j \in F_1^1$, then $w_{n+1}w_{n+2} \leq_p w_i$ and $w_{n+1}w_{n+2} \leq_p w_j$, so $w_{n+1}w_{n+2}\Lambda\ x_4\Lambda\ x_p = w_{n+1}w_{n+3}\Lambda\ y_q$, which contradicts the fact that $a <_p w_{n+3}$ and $b <_p w_{n+2}$. If $x_3 = w_m$, then $A = \{w_n, w_m, w_i, w_j\}$. If $m = n+1$, then $A = \{w_n, w_{n+1}, w_i, w_j\}$. If $m = n-1$, then $A = \{w_{n-1}, w_n, w_i, w_j\}$. Both contradicts condition (1). If $m \leq n-3$, then $w_{n+1} = w_m w_{m+1} w_{m+3}\Lambda$, we have $x_4\Lambda\ x_p = w_{m+1}w_{m+3}\Lambda\ y_2\Lambda\ y_q$. So $x_4 = w_n$. this implies that $w_{m+1}w_{m+2}\Lambda\ x_5\Lambda\ x_p = w_{m+1}w_{m+3}\Lambda\ y_2\Lambda\ y_q$, which contradicts the fact that $b <_p w_{m+3}$ and $a <_p w_{m+2}$. If $m \geq n+3$, then $w_m = w_{n+1}w_{n+2}\Lambda$, this implies that $w_{n+1}w_{n+2}\Lambda\ x_4\Lambda\ x_p = w_{n+1}w_{n+3}\Lambda\ y_q$. Since $a <_p w_{n+3}$ but $b <_p w_{n+2}$, which is a contradiction.

(2-1-3) If $n \geq i+3$, then $x_2 = w_n = w_{i+1}w_{i+2}\Lambda$, we have $w_{i+1}w_{i+2}\Lambda\ x_3\Lambda\ x_p = w_{i+1}w_{i+3}\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$.

(2-2) If $j - i = 2$, then $w_j = w_i w_{i+1}$, we have $x_2\Lambda\ x_p = w_{i+1}y_2\Lambda\ y_q$, so $x_2 = w_n$.

(2-2-1) If $n = i+1$ or $n = i-1$, then $A = \{w_i, w_{i+1}, w_{i+2}, w_m\}$ or $A = \{w_{i-1}, w_i, w_{i+2}, w_m\}$, which contradicts condition (1).

(2-2-2) If $n \leq i-3$, then $w_{i+1} = w_n w_{n+1}\Lambda$ and it must have $x_3\Lambda\ x_p = w_{n+1}w_{n+3}\Lambda\ y_q$. We have a same contradiction like (2-1-2).

(2-2-3) If $n = i+3$, then $A = \{w_i, w_{i+2}, w_{i+3}, w_m\}$. If $m = i+1$ or $m \leq i-1$ or $m \geq i+4$, then $A = \{w_i, w_{i+1}, w_{i+2}, w_{i+3}\}$ or $A = \{w_m, w_i, w_{i+2}, w_{i+3}\}$ or $A = \{w_i, w_{i+2}, w_{i+3}, w_m\}$, we all have contradictions with condition(1), (4), (3).

(2-2-4) If $n = i+5$, then $w_n = w_{i+1}w_{i+2}w_{i+4}$. We have $w_{i+2}w_{i+4}x_3\Lambda\ x_p = y_2\Lambda\ y_q$. So $y_2 = w_i$ or $y_2 = w_j$ or $y_2 = w_m$.

(2-2-4-1) If $y_2 = w_i$, then $w_{i+1}w_{i+4}x_3\Lambda\ x_p = y_3\Lambda\ y_q$. This implies that $y_3 = w_n$, we have $w_{i+1}w_{i+4}x_3\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}y_4\Lambda\ y_q = w_{i+1}w_{i+2}w_{i+3}x_3\Lambda\ x_p$, which contradicts the fact that $a <_p w_{i+4}$ but $b <_p w_{i+3}$.

(2-2-4-2) If $y_2 = w_m$, then $w_{i+2}w_{i+4}x_3\Lambda\ x_p = w_m y_3\Lambda\ y_q$. If $m \geq i+4$, then we have $w_m = w_{i+2}w_{i+3}\Lambda$, so $w_{i+2}w_{i+4}x_3\Lambda\ x_p = w_{i+2}w_{i+3}\Lambda\ y_3\Lambda\ y_q$. Since $a <_p w_{i+4}$ but $b <_p w_{i+3}$, which is a contradiction. If $m \leq i-2$, then we have $w_{i+2} = w_m w_{m+1} w_{m+3}\Lambda$ and $w_{m+1}w_{m+3}\Lambda\ x_p = y_4\Lambda\ y_q$. So $y_4 = w_n$. Hence $w_{m+1}w_{m+3}\Lambda\ x_p = w_{m+1}w_{m+2}\Lambda\ y_q$, which contradicts the fact that $a <_p w_{m+3}$ and $b <_p w_{m+2}$.

(2-2-4-3) If $y_2 = w_j$, then $w_{i+4}x_3\Lambda\ x_p = y_3\Lambda\ y_q$, we obtain that $y_3 = w_i$ or $y_3 = w_j$ or $y_3 = w_m$. If $y_3 = w_i$, then $w_{i+1}w_{i+3}x_3\Lambda\ x_p = y_4\Lambda\ y_q$. This implies that $y_4 = w_n$. Hence we have $w_{i+1}w_{i+3}x_3\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}y_5\Lambda\ y_q$, which contradicts the face that $a <_p w_{i+2}$ and $b <_p w_{i+3}$. If $y_3 = w_j$, then we have $w_{i+3}x_3\Lambda\ x_p = y_4\Lambda\ y_q$. We obtain that $y_4 = w_n$. So

$x_3\Lambda\ x_p = w_{i+4}y_5\Lambda\ y_q$. It follows that $x_3 = w_i$ or $x_3 = w_j$ or $x_3 = w_m$. If $x_3 = w_i$, then $x_4\Lambda\ x_p = w_{i+1}w_{i+3}y_5\Lambda\ y_q$. So $x_4 = w_n$. Since $w_{i+1}w_{i+2}w_{i+4}x_5\Lambda\ x_p = w_{i+1}w_{i+3}y_5\Lambda\ y_q$, , which contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$. If $x_3 = w_m$, then we have $w_m x_4\Lambda\ x_p = w_{i+4}y_5\Lambda\ y_q$. If $m \le i-2$, then $w_{i+4} = w_m w_{m+1}w_{m+3}\Lambda$. It follows that $x_4\Lambda\ x_p$ $= w_{m+1}w_{m+3}\Lambda\ y_5\Lambda\ y_q$. So $x_4 = w_n = w_{m+1}w_{m+2}\Lambda\ w_{i+2}w_{i+4}$. That is, $w_{m+1}w_{m+3}\Lambda\ y_5\Lambda\ y_q =$ $w_{m+1}w_{m+2}\Lambda\ w_{i+2}w_{i+4}x_5\Lambda\ x_p$, which contradicts the fact that $a <_p w_{m+2}$ and $b <_p w_{m+3}$. If $m = i+4$, then $x_4\Lambda\ x_p = y_5\Lambda\ y_q$, which contradicts the minimal length of the word $|w|$. If $m = i+6$, then $w_{i+5}x_4\Lambda\ x_p = y_5\Lambda\ y_q$. So $y_5 = w_n$ and $x_4\Lambda\ x_p = y_6\Lambda\ y_q$, which contradicts the minimal length of the word $|w|$. If $m = i+8$, then $w_m = w_{i+4}w_{i+5}w_{i+7}$. So $w_{i+5}w_{i+7}x_4\Lambda\ x_p = y_5\Lambda\ y_q$. It follows that $y_5 = w_n$. Hence $w_{i+7}x_4\Lambda\ x_p = y_6\Lambda\ y_q$. So $y_6 = w_n$, and $w_{i+6}x_4\Lambda\ x_p = y_7\Lambda\ y_q$. This implies that $y_7 = w_i$ or $y_7 = w_j$ or $y_7 = w_m$. If $y_7 = w_i$, then $w_{i+1}w_{i+3}w_{i+5}x_4\Lambda\ x_p = y_8\Lambda\ y_q$, which implies that $y_8 = w_n = w_{i+1}w_{i+2}w_{i+4}$. So $w_{i+1}w_{i+3}w_{i+5}x_4\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}y_9\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$. If $y_7 = w_j$, then $w_{i+3}w_{i+5}x_4\Lambda\ x_p = y_8\Lambda\ y_q$. So $y_8 = w_n = w_{i+3}w_{i+4}$. Hence we have $w_{i+3}w_{i+5}x_4\Lambda\ x_p = w_{i+3}w_{i+4}y_9\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_7 = w_m$, then $x_4\Lambda\ x_p = w_{i+7}y_8\Lambda\ y_q$. So $x_4 = w_n$. Similar to above, we know that $x_5 = w_m$ and $y_8 = w_n$. Continuing this process, we get that $x_{2k} = y_{2k+2} = w_n$, $x_{2k+1} = y_{2k+3} = w_m$ for any $k \ge 2$. This implies the length of $|w|$ is infinite, which is a contradiction. If $m \ge i+10$, then $w_m = w_{i+4}w_{i+5}w_{i+7}w_{i+9}\Lambda$. So $w_{i+5}w_{i+7}w_{i+9}\Lambda\ x_p = y_5\Lambda\ y_q$. So $w_5 = w_n$ and $w_{i+7}w_{i+9}\Lambda\ x_p = y_6\Lambda\ y_q$. So $y_6 = w_n$ and $w_{i+6}w_{i+9}\Lambda\ x_p = y_7\Lambda\ y_q$. This implies that $y_7 = w_i$ or $y_7 = w_j$ or $y_7 = w_m$. If $y_7 = w_i$, then $w_{i+1}w_{i+3}\Lambda\ x_p = y_8\Lambda\ y_q$. So $y_8 = w_{i+1}w_{i+2}w_{i+4}$, then $w_{i+1}w_{i+3}\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}y_9\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$. If $y_7 = w_j = w_{i+2}$, then $w_{i+3}w_{i+5}\Lambda\ x_p = y_8\Lambda\ y_q$. So $y_8 = w_{i+3}w_{i+4}$, then $w_{i+3}w_{i+5}\Lambda\ x_p = w_{i+3}w_{i+4}y_9\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_7 = w_m$, then $w_{i+6}w_{i+9}\Lambda\ x_p = w_{i+6}w_{i+7}w_{i+9}\Lambda\ y_9\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+8}$ and $b <_p w_{i+9}$. If $y_3 = w_m$, then $w_{i+4}x_3\Lambda\ x_p =$ $w_m y_4\Lambda\ y_q$. So we compare the size of m and $i+4$. The process is the same as $x_3 = w_m$. So we have a contradiction.

(2-2-5) If $n \ge i+7$, then $w_n = w_{i+1}w_{i+2}w_{i+4}w_{i+6}\Lambda$. So $w_{i+2}w_{i+4}w_{i+6}\Lambda\ x_3\Lambda\ x_p = y_2\Lambda\ y_q$. It follows that $y_2 = w_i$ or $y_2 = w_j$ or $y_2 = w_m$. If $y_2 = w_i$, then $w_{i+1}w_{i+4}\Lambda\ x_p = y_3\Lambda\ y_q$, so $y_3 = w_n$ and it must have $w_{i+1}w_{i+4}\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}\Lambda\ y_4\Lambda\ y_q = w_{i+1}w_{i+2}w_{i+3}\Lambda\ x_p$, which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_2 = w_m$ and $m \le i-2$, then $w_{i+2} = w_m w_{m+1}w_{m+3}\Lambda$. We have $w_{m+1}w_{m+3}\Lambda\ x_3\ \Lambda\ x_p = y_3\Lambda\ y_q$. So $y_3 = w_n = w_{m+1}w_{m+2}\Lambda$ and $w_{m+1}w_{m+3}\Lambda\ x_3\Lambda\ x_p = w_{m+1}w_{m+2}\Lambda\ y_4\Lambda\ y_q$, which contradicts the fact that $a <_p w_{m+3}$

and $b <_p w_{m+2}$. If $m \geq i+4$, then $w_m = w_{i+2}w_{i+3}\Lambda$, so $w_{i+2}w_{i+4}x_3\Lambda\ x_p = w_{i+2}w_{i+3}\Lambda\ y_q$, a contradiction. If $y_2 = w_j$, then $w_{i+4}w_{i+6}\Lambda\ x_3\Lambda\ x_p = y_3\Lambda\ y_q$. Hence $y_3 = w_i$ or $y_3 = w_j$ or $y_3 = w_m$. If $y_3 = w_i$, then $w_{i+1}w_{i+3}w_{i+6}\Lambda\ x_3\Lambda\ x_p = y_4\Lambda\ y_q$, so $y_4 = w_n = w_{i+1}w_{i+2}\Lambda$, which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_3 = w_j$, then we have $w_{i+3}w_{i+6}\Lambda\ x_3\Lambda\ x_p = y_4\Lambda\ y_q$, so $y_4 = w_n$ and $w_{i+3}w_{i+6}\Lambda\ x_3\Lambda\ x_p = w_{i+3}w_{i+4}w_{i+6}\Lambda\ y_5\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+6}$ and $b <_p w_{i+5}$. If $y_3 = w_m$ and $m \leq i-2$, then $w_{i+4} = w_m w_{m+1}w_{m+2}\Lambda$. So $w_{m+1}w_{m+3}\Lambda\ x_3\Lambda\ x_p = y_4\Lambda\ y_q$. We obtain $y_4 = w_n = w_{m+1}w_{m+2}\Lambda$, then $w_{m+1}w_{m+3}\Lambda\ x_3\Lambda\ x_p = w_{m+1}w_{m+2}\Lambda\ y_5\Lambda\ y_q$, which contradicts the fact that $a <_p w_{m+3}$ and $b <_p w_{m+2}$. If $m \geq i+6$, then $w_m = w_{i+4}w_{i+5}\Lambda$, we have $w_{i+4}w_{i+6}\Lambda\ x_p = w_{i+4}w_{i+5}\Lambda\ y_q$, which contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+5}$. If $m = i+4$, then we have $w_{i+6}\Lambda\ x_3\Lambda\ x_p = y_4\Lambda\ y_q$. So $y_4 = w_i$ or $y_4 = w_j$ or $y_4 = w_m$. If $y_4 = w_i$ or $y_4 = w_j$, then $w_{i+1}w_{i+3}w_{i+5}\Lambda\ x_3\Lambda\ x_p = y_5\Lambda\ y_q$ or $w_{i+3}w_{i+5}\Lambda\ x_3\Lambda\ x_p = y_5\Lambda\ y_q$, so $y_5 = w_n$. Hence $w_{i+1}w_{i+3}w_{i+5}\Lambda\ x_p = w_{i+1}w_{i+2}w_{i+4}\Lambda\ y_q$ or $w_{i+3}w_{i+5}\Lambda\ x_p = w_{i+3}w_{i+4}\Lambda\ y_q$, the first contradicts the fact that $a <_p w_{i+2}$ and $b <_p w_{i+3}$, the second contradicts the fact that $a <_p w_{i+4}$ and $b <_p w_{i+3}$. If $y_4 = w_m$ and $n \geq i+9$, then $y_5\Lambda\ y_q = w_{i+5}w_{i+8}\Lambda\ x_3\Lambda\ x_p$, so $y_5 = w_n = w_{i+5}w_{i+6}w_{i+8}\Lambda$, and we have $w_{i+5}w_{i+6}w_{i+8}\Lambda\ y_6\Lambda\ y_q = w_{i+5}w_{i+8}\Lambda\ x_3\Lambda\ x_p$, since $a <_p w_{i+6}$ but $b <_p w_{i+7}$, which is a contradiction. If $n = i+7$, then $y_5\Lambda\ y_q = w_{i+5}x_3\Lambda\ x_p$. It follows that $y_5 = w_n = w_{i+5}w_{i+6}$, so we obtain that $x_3 = w_i$ or $x_3 = w_j$ or $x_3 = w_m$. If $x_3 = w_i$ or $x_3 = w_m$, we have same contradictions as above. So $x_3 = w_m$. Following this process, we know $x_s = y_s = w_m$, $x_{s+1} = y_{s+1} = w_n$ for $s \geq 3$ and s is an odd number, which indicates that the length of $|w|$ is infinite. It is a contradiction.

Acknowledgments: The authors would like to thank the anonymous referees for their careful reading of the manuscript and useful suggestions.

References

[1]. H.J. Shyr (2001), Free Monoids and Languages. 3rd ed. *Hon Min Book Company*, Taichung, Taiwan.

[2].S.S Yu and Yu-Kuang Zhao (2000), “Properties of Fibonacci languages”, Discrete Mathematics, 224 (1–3): 215-223.

[3]. Z.Z. Li, Y.S. Tsai and G.C. Yih (2004), “Characterizations on codes with three elements”, *Soochow Journal of Mathematics*, 30(2): 177-196.