Computer Communication & Collaboration (2013)
Submitted on 24/May/2013
Article ID: 2292-1028-2013-01-32-12

Some Properties of p−Primitive Words[1] *

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Abstract: In this paper, we first give a sufficient condition to judge a homomorphism which preserves p−primitive words. Although it was concerned by Huang C.C. and Yu S.S. in 2008 in *Discrete Mathematics*, our result is an easier method than theirs. Then we construct two kinds of p−primitive words. At last we prove the set of all p−primitive words with even lengths and odd lengths are disjunctive languages.

Keywords: *p*−primitive word, homomorphism, comma-free code, infix code, disjunctive language

2010 Mathematics Subject Classification: 68Q45; 68Q70

1 Introduction

Primitive word is a very fundamental concept and is an important research topic in formal language theory, and primitive words play a considerable role in the field of combinatorics on words, an area in discrete mathematics motivated in part by computer science, information theory, coding theory and molecular biology (see [4,7,10,15]). The study of homomorphisms preserving words and languages is motivated by considerations of inheritance properties for generating systems. Such properties are very important characters of the data structures in an artificial intelligence

[1] This work is supported by National Natural Science Foundation of China # 11261066,Applied Basic Research Programs of Science and Technology Department Foundation of Yunnan Province of China # 2010CD21 and Educational Commission Important Project of Yunnan Province of China # 2011Z008.

system. On the other hand, homomorphisms preserving words and languages are also related to OL schemes which are parallel rewriting systems originally introduced in 1968 to model the development of the multicellular organisms in Bioinformatics (see [7,8]). Homomorphisms preserving primitive words, $p-$primitive words were well concerned in [1-7].

Some definitions and lemmas are quoted in section 2. In section 3, we focus on homomorphism preserving $p-$primitive words and obtain the following result:

Let h be a homomorphism from X^* into X^*. If $h(X)$ is a comma-free code and $h(X)$ is a $p-$primitive word for each $p-$primitive word x with $\lg(x)\leq 2$, then h preserves all $p-$primitive words. Homomorphism preserving $p-$primitive words was studied in [1] and the following result was obtained:

A homomorphism h of X^* is $p-$primitive words preserving if and only if h satisfies the following properties:

(i) $h(w)$ is $p-$primitive for any $p-$primitive word w with $\lg(w)\leq 3$;

(ii) there exists no $a\in X$ such that $h(a)\in x\operatorname{Pre}f(h(w))h(v)x$ for any $v\in P_1\cup\{1\}$, $w\in X^+\setminus vaX^*$ and $x\in X*$;

(iii) there exists no $p-$primitive word $w\in X^+$ such that $xh(w)xy\in h(x)$ for any $x\in X^+$ and $y\in X^*$;

(iv) if there exists a word x with $x<_p h(v)$ and $x<_p h(b)$ for some $v\in X^+$ and $b\in X$ such that $(x^{-1}h(v))h(u)(h(b)x^{-1})\leq_p h(w)$ for some $u,w\in X^*$, then $vubw$ is not a $p-$primitive word. It can be seen that conditions in our result is much easier and better to judge than that in [1].

In [9], $abmx$ and ab^mxab^my were proved to be $p-$primitive words for some different letters a,b and different words x,y with $\lg(x)=\lg(y)=k$ for any $m\geq k$. In section 4, we will show that the following two kinds of words are also $p-$primitive words:

For two different letters a,b and different non-empty words x,y such that x is not a prefix of y, the word $ab^m xab^m y$ is a $p-$primitive word for any $m \geq k$ where $k = \max\{\lg(x), \lg(y)\}$;

For two different letters a,b and different words $x_1, x_2 \ldots x_n$, whose lengths are equal to k, the word $ab^m x_1 ab^m x_2 \ldots ab^m x_n$ is a $p-$primitive word for any $m \geq k$.

Regular languages and disjunctive languages play an important role in the field of theoretical computer science. In section 5, we prove that the set of all $p-$primitive words with even lengths and the set of all $p-$primitive words with odd lengths are disjunctive languages. This is a good complement to the statement of the set of all primitive words with even lengths and odd lengths are disjunctive languages (see [4]).

2 Preliminaries

Let X be a nonempty finite set of letters, which is called an alphabet. Any finite string over X is called a word. For example, $w = bababbaaa$ is a word over $X = \{a,b\}$. The word that contains no letter is called the *empty word*, denoted by 1. The set of all words is denoted by X^*, which is a free monoid with concatenation. For example, the product of two words $x = babb$ and $abbbaa$ is the word $xy = babbabbbaa$. Let $X^+ = X^* \setminus \{1\}$. For any word w in X^+, let $\lg(w)$ be the length of w which is the number of letters that occur in w and $\lg(1) = 0$. Then $\lg(w) = 9$ for the former word $w = bababbaaa$. Let $X^k = \{x \in X^* \mid \lg(x) = k\}$, where k ≥ $k \geq 0$. Let $u, x \in X^+$, we call u is a prefix of v and denote by $u \leq_p v$ if $v = ux$ for some $x \in X^*$. If $u, v \in X^+$, then u is a proper prefix of v and denote by $u <_p v$. Let $pre(v) = \{u \in X^* \mid u \leq_p v\}$. Similarly, $u \leq_s v$ and $u <_s v$ are defined.

For any $w \in X^+$, let $w^0 = 1$, we call a $n+1$ power of w is $w^{n+1} = w^n w$ for $n \geq 0$. For instance, if $w = aaa$ where $a \in X$, we call it the fourth power of a and its length is 4. A nonempty word which is not a power of any other nonempty word is called a primitive word. A nonempty word which is not beginning with a square of any other nonempty word is called a prefix primitive word shortly p −primitive word. Every p −primitive word is a primitive word (see [1,2,4]).

Any non-empty subset of X^+ is called a language. A language P is called a *prefix code* if $P \cap PX^+ = \varnothing$. A language L is called an *infix code* if $L \cap X^+L = \varnothing$. A language L is called a *comma-free code* if $L^2 \cap X^+LX^+ = \varnothing$.

Lemma 2.1[4] If L is a comma-free code, then L is an infix code.

A mapping h from X^* into X^* such that $h(xy) = h(x)h(y)$ for all $x, y \in X^*$ is called a homomorphism. A homomorphism of X^* is called non-erasing if $h(a) \neq 1$ for each $a \in X$. Let $h: X^* \to X^*$ be a homomorphism and F be a set of some languages or some words. If $h(A) \in F$ for every $A \in F$, then we say that h preserves F or that h is F-preserving. For example, let $F =$ {primitive words on X }. Then a homomorphism h preserves primitive words if $h(w)$ is a primitive word for every primitive word $w \in X^+$.

Definitions and items which are used in the paper but not stated here can be found in [4,10]. In this paper, we always let $|X \geq 2|$.

3 p −Primitive Words Preserving Homomorphism

Lemma 3.1 Let $h: X^* \to X^*$ be a homomorphism. If $h(x)$ is p −primitive for each p −primitive word x with $\lg(x) \leq 2$, then h is a non-erasing injective homomorphism and $h(X)$ is a prefix code.

Proof. For any $a \in X$, since a is a p −primitive word, then $h(a)$ is a p −primitive word. Thus h is non-erasing. Suppose h is not injective. Then there exist $w, z \in X^+$ and $w \neq z$ such that $h(w) = h(z)$. Let $w = a_1a_2 \cdots a_n$ and $z = b_1b_2 \cdots b_m$ where $a_i, b_j \in X$ and

$i=1,2\cdots,n$, $j=1,2\cdots,m$. We may let $a_1 \neq b_1$. Since $h(w)=h(z)$, then $h(a_1)h(a_2)\cdots h(a_n)=h(b_1)h(b_2)\cdots h(b_n)$. If $\lg(h(a_1)) \leq \lg(h(b_1))$, then $h(b_1)=h(a_1)g$ for some $g \in X^*$. Thus $h(a_1b_1)=[h(a_1)]^2 g$, which contradicts with $h(a_1b_1)$ is p−primitive for the p−primitive word a_1b_1 with $\lg(a_1b_1)=2$. If $\lg(h(a_1)) > \lg(h(b_1))$, then we also have a contradiction. Thus h is injective. Suppose that $h(X)$ is not a prefix code, then there exist $a,b \in X$ and $a \neq b$ such that $h(a) \leq_p h(b)$. Then $h(b)=h(a)y$ for some $y \in X^+$. Hence $h(ab)=(h(a))^2 y$, which contradicts $h(ab)$ is a p−primitive word for the p−primitive word ab with $\lg(ab)=2$.

Theorem 3.2 Let $h: X^* \to X^*$ be a homomorphism. If $h(X)$ is a comma-free code and $h(w)$ is a p−primitive word for each $w \in X^*$ with $\lg(w) \leq 2$, then h preserves all p−primitive words.

Proof. Suppose h is not a p−primitive words preserving homomorphism. Then there exists a p−primitive word $w=w_1w_2\cdots w_n$ where $w_i \in X$, $i=1,2,\cdots,n$ and $n \geq 3$ such that $h(w)$ is not a p−primitive word. Then $h(w)=u^2v$ for some $u \in X^+$ and $v \in X^*$. That is $h(w_1)\cdots h(w_n)=u^2v$. Hence $u=h(w_1w_2\cdots w_{i-1})y_1$, $h(w_i)=y_1y_2$, and $y_2h(w_{i+1}\cdots w_n)=uv$ for some $y_1, y_2 \in X^*$ and $1 \leq i \leq n$.

If $y_1=1$, then $u=h(w_1w_2\cdots w_{i-1})$. Since $u \in X^*$, then $2 \leq i \leq n-1$ at this case. Then $h(w_i)\cdots h(w_n)=h(w_1)x_1$. If $\lg(h(w_i)) > \lg(h(w_1))$, then there exists $x_1 \in X^+$ such that $h(w_i)=h(w_1)x_1$, which contradicts $h(X)$ is a prefix code by Lemma 3.1. So does the case $\lg(h(w_{i+1})) < \lg(h(w_1))$. If $\lg(h(w_i))=\lg(h(w_1))$, then $h(w_i)=h(w_1)$. Since h is injective by Lemma 3.1, then $w_i=w_1$. Similarly, we have $w_2=w_{i+1},\cdots w_{i-1}=w_{2i-1}$. Thus

$w = w_1 w_2 \cdots w_n = (w_1 w_2 \cdots w_i)^2 w_{2i} \cdots w_n$, which contradicts w is a p−primitive word.

Thus $y_1 \neq 1$. Similarly we get $y_2 \neq 1$.

Let y_1 and y_2 are non-empty words. Then $u = h(w_1 w_2 \cdots w_{n-1}) y_1, h(w_n) = y_1 y_2$,

$y_2 h(w_{i+1}) \cdots h(w_n) = uv = h(w_1) \cdots h(w_{i-1}) y_1 v$ where $y_1, y_2 \in X^+$,and $1 \leq i \leq n$. We consider the following cases:

(1) If $i = n$, then $u = h(w_1 w_2 \cdots w_{n-1}) y_1$, $h(w_n) = y_1 y_2$ and $y_2 = uv$. Hence $h(w_2) = y_1 uv$.

So $h(w_n) = y_1 h(w_1 \cdots w_{n-1}) y_1 v = y_1 h(w_1)[h(w_2 \cdots w_{n-1}) y_1 v]$, which contradicts $h(X)$ is a comma-free code such that it is an infix code by Lemma 2.1.

(2) If $2 \leq i \leq n-1$, then $u = h(w_1 w_2 \cdots w_{i-1}) y_1$, $h(w_i) = y_1 y_2$ and

$y_2 h(w_{i+1} \cdots w_n) = uv = h(w_1) \cdots h(w_{i-1}) y_1 v.$ We consider the following cases:

(2-1) If $\lg(y_2) > \lg(h(w_1))$, then there exists $z_1 \in X^+$ such that $y_2 = h(w_1) z_1$. So

$h(w_i) = y_1 y_2 = y_1 h(w_1) z_1$, which contradicts $h(X)$) is an infix code by Lemma 2.1.

(2-2) If $\lg(y_2) \leq \lg(h(w_1))$, then there exists $z_2 \in X^*$ such that $h(w_1) = y_2 z_2$. So

$h(w_{i+1}) \cdots h(w_n) = z_2 h(w_2 \cdots w_{i-1}) y_1 v$.

(2-2-1) If $\lg(z_2) > \lg(h(w_{i+1}))$, then there exists $z_3 \in X^+$ such that $z_2 = h(w_{i+1}) z_3$.

Then $h(w_1) = y_2 z_2 = y_2 h(w_{i+1}) z_3$, which contradicts $h(X)$ is an infix code by Lemma 2.1.

(2-2-2) If $\lg(z_2) < \lg(h(w_{i+1}))$, then there exists $z_4 \in X^+$ such that $h(w_{i+1}) = z_2 z_4$.

Since $h(w_i) = y_1 y_2$, then $\lg(h(w_i w_{i+1})) = y_1 y_2 z_2 z_4 = y_1 h(w_1) z_4$, which contradicts $h(X)$ is a comma-free code.

(2-2-3) If $\lg(z_2) = \lg(h(w_{i+1}))$, then $z_2 = h(w_{i+1})$. Therefore

$h(w_1) = y_2 z_2 = y_2 h(w_{i+1})$, which contradicts $h(X)$ is an infix code by Lemma 2.1.

(3) If $i = 1$, then $u = y_1$, $h(w_1) = y_1 y_2$ and $y_2 h(w_2 \cdots w_n) = uv$. So $h(w_1) = uy_2$ and

$y_2 h(w_2) \cdots h(w_n) = uv$.

(3-1) If $\lg(y_2) \geq \lg(u)$, then there exists $u_1 \in X^*$ such that $y_2 = uu_1$. So $h(w_1) = uy_2 = uuu_1 = u^2 u_1$, which contradicts $h(w_1)$ is a p−primitive word for the p−primitive word w_1 with $\lg(w_1) = 1$.

(3-2) If $\lg(y_2) < \lg(u)$, then there exists $u_2 \in X^+$ such that $u = y_2 u_2$. Then $y_2 h(w_2) \cdots h(w_n) = uv = y_2 u_2 v$. Thus $h(w_2) \cdots h(w_n) = u_2 v$.

(3-2-1) If $\lg(u_2) \leq \lg(h(w_2))$, then there exists $u_3 \in X^*$ such that $h(w_2) = u_2 u_3$. Hence $h(w_1 w_2) = uy_2 u_2 u_3 = u^2 u_3$ is not a p−primitive word. However, since $w = w_1 w_2 \cdots w_n$ is a p−primitive word, then $w_1 \neq w_2$. Since h preserves each p−primitive word w with $\lg(w) = 2$, then $h(w_1 w_2)$ is a p−primitive word. This is a contradiction.

(3-2-2) If $\lg(u_2) > \lg(h(w_2))$, then there exists $u_4 \in X^+$ such that $u_2 = h(w_2) u_4$. Hence $h(w_1) = uy_2 = y_2 u_2 y_2 = y_2 h(w_2) u_4 y_2$, which contradicts $h(X)$ is an infix code by Lemma 2.1.

So no matter $y_1 = 1$ or $y_1 \neq 1$, we all have contradictions. Thus, if $h(X)$ is a comma-free code and $h(x)$ is a p−primitive word for any p−primitive word w with $\lg(w) \leq 2$, then h preserves all p−primitive words.

4 Some Kinds of p−Primitive Words

Lemma 4.1[9] Let $a, b \in X$, $a \neq b$ and $x_1, x_2 \in X^+$, $x_1 \neq x_2$ such that $\lg(x_1) = \lg(x_2) = k \geq 1$. Then $ab^m x_1$ and $ab^m x_1 ab^m x_2$ are p−primitive words for every $m \geq k$.

Theorem 4.2 Let $a, b \in X$, $a \neq b$ and $x_1, x_2 \in X^+$, $x_1 \neq x_2$ such that x_1 is not a prefix of x_2. Then $ab^m x_1 ab^m x_2$ is a p−primitive word for every $m \ge k$ where

$k = \max\{\lg(x_1), \lg(x_2)\}$.

Proof. Suppose that $ab^m x_1 ab^m x_2$ isn't a p−primitive word. Then there exists $v \in X^+$ such that $v^2 \le_p ab^m x_1 ab^m x_2$. So $\frac{m+1+\lg(x_1)}{2} < \lg(v) \le m+1+\frac{\lg(x_1)+\lg(x_2)}{2}$. If $\frac{m+1+\lg(x_1)}{2} < \lg(v) < m+1$, then $v = ab^t = b^{m-t} y_1$ for some $y_1 \in X^*$ and $t \ge 0$. Thus $a = b$, which is a contradiction. We consider the following cases:

(1) If $\lg(x_1) > \lg(x_2)$ and $m+1 < \lg(v) \le m+1+\frac{\lg(x_1)+\lg(x_2)}{2} < m+1+\lg(x_1)$, then $v = ab^m y_2 = y_3 ab^{t'} y_4$ where $x_1 = y_2 y_3$, $y_2, y_4 \in X^*, y_3 \in X^+$ and $0 \le t' \le m$. Since $\lg(y_3) \le \lg(x_1) \le m$, then comparing the $(\lg(y_3)+1)th$ letter in v, we have $a = b$. This is also a contradiction.

(2) If $\lg(x_1) < \lg(x_2)$, then we consider the following cases:

(2-1) If $1+m \le \lg(v) < m+1+\lg(x_1)$, then $v = ab^m z_1 = z_2 ab^{s_2} y$ where $x = z_1 z_2$, $z_1, y \in X^*, z_2 \in X^+$ and $0 \le s_2 \le m$. Since $\lg(z_2) < \lg(x_1) \le m$, comparing the $(\lg(x_2)+1)th$ letter in v, we have $a = b$. This is also a contradiction.

(2-2) If $\lg(v) = m+1+\lg(x_1)$, then $v = ab^m x_1 = ab^m z_3$, where $z_3 \le_p x_2$. Then $x_1 = z_3 \le_p x_2$, which is a contradiction.

(2-3) If $m+1+\lg(x_1) < \lg(v) \le m+1+\frac{\lg(x_1)+\lg(x_2)}{2} = m+1+\lg(x_1)+\frac{\lg(x_2)-\lg(x_1)}{2}$, then $v = ab^m x_1 ab^{t''} = b^{m-t''} z_4$ for some $z_4 \in X^+$ and $0 \le t'' < m$, because $\frac{\lg(x_2)-\lg(x_1)}{2} < m$. Comparing the first letter in v, we have $a = b$. This is also a contradiction.

From all of the above and Lemma 4.1, we know that $ab^m x_1 ab^m x_2$ is a p−primitive word .

In the following , we will show Lemma 4.1 is right for $n \ge 3$. First a Lemma is quoted for use.

Lemma 4.3[9] Let $f, g \in X^*$, $\lg(f) = \lg(g)$, $f = g$. If $g, fg \in Q_p$, then $fg^i \in Q_p$ for any $i \geq 2$. By Lemmas 4.1 and 4.4, we know if $a, b \in X, a \neq b$ and $x, y \in X^+$ such that $\lg(y) = \lg(x) = k$, then $ab^m x(ab^m y)^i \in Q_p$, for every $m \geq k$ and $i \geq 2$.

Theorem 4.4 Let $a, b \in X, a \neq b$ and $x_1 x_2 \cdots x_n \in X^k$ where $k \geq 1, x_i \neq x_j$ when $i \neq j$ and $i, j = 1,2,\cdots,n$. Then $ab^m x_1 ab^m x_2 \cdots ab^m x_n \in Q_p$ for every $m \geq k$.

Proof. By Lemma 4.1, we know $ab^m x_1, ab^m x_1 ab^m x_2 \in Q_p$. Suppose the theorem is right for the integer $n-1$. We will show it is right for n by induction. Suppose that $ab^m x_1 ab^m x_2 \cdots ab^m x_{n-1} ab^m x_n$ isn't a p−primitive word, then there exists a non-empty word w such that $w^2 \leq_p ab^m x_1 ab^m x_2 \cdots ab^m x_{n-1} ab^m x_n$.

(1) If n is odd, then $\frac{n-1}{2}(1+m+k) < \lg(w) \leq \frac{n-1}{2}(1+m+k) + \frac{1+m+k}{2}$. Let $w = ab^m x_1 \cdots ab^m x_{\frac{n-1}{2}} ab^t = b^{m-t} x_{\frac{n+1}{2}} x_{\frac{n+3}{2}} \cdots$, where $0 < \lg(ab^t) \leq \frac{1+m+k}{2}$. Then $0 \leq t \leq \frac{1+m+k}{2} - 1 = \frac{m+k-1}{2}$. Thus $0 < \frac{1}{2} \leq \frac{1+m-k}{2} \leq m-t \leq m$. Comparing the first letter in w, we have $a = b$. This is a contradiction.

(2) If n is even, then $(\frac{n}{2}-1)(1+m+k) + \frac{1+m+k}{2} < \lg(w) \leq \frac{n}{2}(1+m+k)$.

(2-1) If $(\frac{n}{2}-1)(1+m+k) + \frac{1+m+k}{2} < \lg(w) \leq (\frac{n}{2}-1)(1+m+k) + 1 + m$, then $w = ab^m x_1 \cdots ab^m x_{\frac{n}{2}-1} ab^s = b^{m-s} ab^m \cdots$, where $\frac{1+m+k}{2} < \lg(ab^s) < 1+m$. Hence $\frac{1+m+k}{2} < s < m$. Comparing the first letter in w, we have $a = b$, which is a contradiction.

(2-2) If $(\frac{n}{2}-1)(1+m+k) + 1 + m \leq \lg(w) \leq (\frac{n}{2}-1)(1+m+k) + k + m$, then $w = ab^m x_1 \cdots ab^m x_{\frac{n}{2}-1} ab^m y_1 = y_2 ab^m x_{\frac{n+1}{2}} \cdots$, where $x_{\frac{n}{2}} = y_1 y_2$ and $y_1, y_2 \in X^*$. Then $1 + m \leq \lg(ab^m y_1) \leq k + m$, so $1 \leq \lg(y_1) \leq k-1$. Therefore $1 \leq \lg(y_2) \leq k$. Then we have

$2 \le \lg(y_2 a) \le k+1 \le m+1$. Comparing the $(\lg(y_2)+1)$ th letter in w, we have $a=b$. This is a contradiction.

(2-3) If $\lg(w)=\frac{n}{2}(1+m+k)$, then $w=ab^m x_1 \cdots ab^m x_{\frac{n}{2}} = ab^m x_{\frac{n+1}{2}} \cdots ab^m x_n$. Hence $x_1 = x_{\frac{n}{2}+1}, \cdots, x_{\frac{n}{2}} = x_n$, which contradicts $x_i \ne x_i$ when $i \ne j$.

Thus from all above we have $ab^m x_1 ab^m x_2 \cdots ab^m x_n \in Q_p$.

5 Disjunctive Languages Related to p−Primitive Words

Let A be a language and for $x, y \in X^*$, $x \equiv y\ (P_A)$ if $uxv \in A$ if and only if $uyv \in A$ for all $u, v \in X^*$. The equivalent relation P_A is a congruence and called *the principal congruence determined by* A. If there are finite P_A classes then L is called a *regular language*. If every P_A class contains only one word, that is P_A is the equality then A is called a *disjunctive language*. The following statements are important methods to judge a language is disjunctive.

Lemma 5.1[4] Let L be a language. Then the following three statements are equivalent:

(1) L is a disjunctive language;

(2) Let $x, y \in X^*$ and $\lg(x)=\lg(y)$, if $x \equiv y\ (P_L)$, then $x=y$;

(3) For all $x, y \in X^*$, there exist $u, v \in X^*$ such that $uxy \in L$ and $uyv \notin L$ or vice versa.

Let $Q_{p_{ev}} = \{ f \in Q_p \mid \lg(f) \text{ if even} \}$ and $Q_{p_{od}} = Q \setminus Q_{p_{ev}} = \{ f \in Q_p \mid \lg(f) \text{ is odd} \}$.

Theorem 5.2 The languages $Q_{p_{ev}}$ and $Q_{p_{od}}$ are disjunctive languages.

Proof. Let $x, y \in X^k$ and $x \ne y$. Let $u = ab^k, v = ab^k y$, then $uxv = ab^k x ab^k y \in Q_p$ by Lemma 4.1, and $uyv = (ab^k y)^2 \notin Q_p$. Since $\lg(x)=\lg(y)=k$, then $\lg(uxv) = 2\lg(ab^k x)$ is even. Therefore $uxv \in Q_{p_{ev}}$. But $uyv \in Q_{p_{ev}}$. Hence $x \ne y\ (P_{Q_{p_{ev}}})$. Thus $Q_{p_{ev}}$ is a disjunctive language by Lemma 5.1.

Let $u = ab^k, v = (ab^k y)^2$, then $uxz = ab^k x(ab^k y)^2 \in Q_p$ by Lemmas 4.1and 4.3, it can be seen that $uyz = (ab^k y)^3 \notin Q_p$. Since $\lg(ab^k) = 1 + 2k$ is odd and $\lg(x) = \lg(y)$, then $\lg(uxz) = 3\lg(ab^k x) = 2(3k+1)+1$ must be odd. Hence $uxz \in Q_{p_{od}}$, $uyz \notin Q_{p_{od}}$. So $x \neq y\ (P_{Q_{pod}})$. Thus $Q_{p_{od}}$ is a disjunctive language by Lemma 5.1.

For any $i \geq 2$, let $Q_p^i = \{ f_1 f_2 \mathbf{L}\ f_i \mid f_j \in Q_p, j = 1,2 \mathbf{L}, n\}$ and $Q_p^{(i)} = \{ f^i \mid f \in Q_p \}$. Then $Q_p^{(i)} \subset Q_p^i$, and $Q_p^i \cap Q \neq \emptyset$ by Lemma 4.3 and Theorem 4.4. Of cause, $Q_p^i \cap Q \neq Q$ and $Q_p^i \cap Q \neq Q_p^i$.

Proposition 5.3 The language $Q_p^i \cap Q$ is a disjunctive language for any $i \geq 2$.

Proof. For any $x, y \in X^k$ where $k > 0$, $x \neq y$, let $u = ab^k$, $v = (ab^k y)^{i-1}$, then $ab^k x \in Q_p$, $ab^k y \in Q_p$. So $uxv = ab^k x(ab^k y)^{i-1} \in Q_p$. By lemma 4.3, we have $uxv \in Q_p \subseteq Q$. Then $uxv \in Q_p^i \cap Q$. Since $uyv = ab^k y(ab^k y)^{i-1} = (ab^k y)^i \notin Q$, we have $uyv \notin Q_p^i \cap Q$. Hence $x \neq y\ (P_{Q_p^i \cap Q})$. Thus $Q_p^i \cap Q$ is a disjunctive language for any $i \geq 2$.

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