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# Multicriteria Evaluation and Optimization
# Nonlinear Trade-off Scheme

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**Abstract:** The concept of nonlinear trade-off scheme in multicriteria problems of evaluation and optimization is presented. It is shown that the problem is to approximate correctly the decision-maker's utility function and construct a substantial mathematical model (scalar convolution) adequate to the given situation to solve various multicriteria problems.

**Keywords:** Multicriteria problems, Situation of decision-making, Adaptation, Model of utility function, Nonlinear compromise scheme

## 1. Problem Description

Finding a multicriteria solution is inherently a compromise and is based on using subjective information. Given this information and a compromise scheme selected, it is possible to pass from the general vector expression to the scalar convolution of partial criteria, which provides a basis for a constructive apparatus to solve multicriteria problems. Solving the problem is based on the hypothesis that there exists a utility function [1] appearing in the DM's brain during the solution of a specific multicriteria problem. We may state that virtually all the approaches to determining the scalar convolution of criteria are reduced to constructing one mathematical model or another of the DM's utility function.

The problem is to approximate correctly the utility function and to construct a substantial mathematical model as a scalar convolution, adequate to the given situation, to solve different multicriteria problems.

## 2. Formalization of the Problem

A DM's utility function can generally be represented as $\Phi[y(x), r]$, where $x = \{x_i\}_{i=1}^n \in X$ is the vector of solutions defined in the feasible domain *X*; $y = \{y_{0k}\}_{k=1}^s \in M$ is the vector of normalized partial criteria defined in the domain $M = \{y \mid 0 \le y_{0k} \le 1, k \in [1, s]\}$; $A = \{A_k\}_{k=1}^s$ is the constraint vector; and $r \in R$ is the vector of external conditions defined on the set of feasible factors ***R***.

The situation of making a multicriteria decision is defined by the factors of external conditions *r*. In solving multicriteria problems, it is usually assumed that the vector *r* is fixed and specified: $r = r^{\mathbf{0}}$. Then the DM's utility function can be represented by the scalar convolution of criteria

$$\Phi[y_0(x), r]_{|r=r^{\mathbf{0}}} = Y[y_0(x)]^{\mathbf{o}},$$

where $Y[y_0(x)]^{\circ}$ is the scalar convolution constructed from the compromise scheme adequate to the given situation.

In most cases, solving multicriteria problems is restricted to a linearized model.

Though such an approach has a doubtless advantage (simplicity), it is characterized by shortcomings inherent in the linearization method. In practical multicriteria problems, it is expedient to construct a ***nonlinear*** model of the DM's utility function (the concept of a nonlinear compromise scheme [2]).

## 3. Conceptual Analysis of the DM's Utility Function

In what follows, we will consider an optimization problem and assume for definiteness that all the criteria $y_0(x)$ are to be ***minimized***. Then mathematically, the vector optimization problem can be represented as

$$x^* = \arg\min_{x \in X} Y[y_0(x)].$$

Let us introduce the concept of the intensity of a situation as a measure of how normalized relative partial criteria are close to the limit value (unity):

$$r_k = 1 - y_{0k},\ r_k \in [0;1],\ k \in [1, s].$$

If a multicriteria decision is made in an intense situation, then in the conditions specified, one or several partial criteria may appear dangerously close to the limit values ( $r_k \approx 0$ ). And if one of the criteria achieves the limit (or is outside it), this event will not be compensated by a possible small level of other criteria (violating any of the constraints is usually prohibited).

In such a situation, it is necessary to interfere (in every possible way) the dangerous increase of the most adverse (i.e., the closest to the limit) partial criterion irrespective of the behavior of other criteria. And in the first polar case ($r_k$=0), the DM leaves only this unique, most unfavorable partial criterion for consideration and neglects the others. Hence, a minimax Chebyshev model (egalitarian principle)

$$x^* = \arg\min_{x \in X} Y[y_0(x)]^{(1)} = \arg\min_{x \in X}\max_{k \in [1,s]} y_{0k}(x)$$

adequately expresses the compromise scheme in an intense situation.

In the second polar case ($r_k$≈1), the situation is quiet, partial criteria are small, and there is no threat to violate the constraints. The DM considers that a unit deterioration of any partial criterion is compensated quite well by an equivalent unit improvement of any other criterion. Such a scheme can be expressed by the model of integral optimality (utilitarian principle)

$$x^* = \arg\min_{x \in X} Y[y_0(x)]^{(2)} = \arg\min_{x \in X} \sum_{k=1}^{s} y_{0k}(x).$$

If we take the conclusions from this analysis as a logic basis for formalizing [3] the choice of a compromise scheme, we can present various constructive concepts such as the concept of a nonlinear compromise scheme.

## 4. Nonlinear Compromise Scheme

From the formalization standpoint, it is expedient to replace the problem of choosing a compromise scheme with the equivalent problem of synthesis of a unified scalar convolution of partial criteria which would express different principles of optimality in different situations.

Thus, a universal convolution should express a compromise scheme adaptable to a situation. We may say that adaptation and adaptability are the main substantial essence of studying multicriteria systems. The scalar convolution should include the explicit characteristics of the situation intensity $r$.

Among the possible functions meeting the above requirements, let us consider an elementary one

$$Y(a, y_0) = \sum_{k=1}^{s} a_k [1 - y_{0k}(x)]^{-1}; a \in \Gamma_a , \quad (1)$$

where $a_k$=const are formal parameters defined on a simplex and having double physical meaning. On the one hand, these are weight coefficients that express the DM's preferences in partial criteria, and on the other hand, these are coefficients of a substantial regression model of the DM's utility function on the concept of a nonlinear compromise scheme.

Thus, a nonlinear compromise scheme is associated with a vector optimization model, which explicitly depends on the characteristics of the situation intensity $r$:

$$x^* = \arg\min_{x \in X} \sum_{k=1}^{s} a_k [1 - y_{0k}(x)]^{-1}. \quad (2)$$

In contrast to the linear model, defined in a small neighborhood of a working point, the nonlinear model is defined on the whole feasible region $X$ and does not require coefficients $a_k$ to be recalculated if the situation varies.

As is seen from the formula, if any relative partial criterion, for example, $y_{0i}(x)$, approaches the limit (unity), i.e., the situation becomes intense, the corresponding term $Y_i = 1/a_i[1 - y_{0i}(x)]$ in the sum being minimized increases so that the minimization of the whole sum reduces to the minimization of only this worst term, i.e., of the criterion $y_{0i}(x)$. And this is a minimax model manifestation.

If relative partial criteria are far from unity, i.e., the situation is quiet, the proposed model operates equivalent to the integral optimality model. In intermediate situations, different degrees of partial alignment of criteria are obtained. Therefore, the nonlinear compromise scheme has the property of continuous adaptation to the situation of making a multicriteria decision.

In optimization problems, the convolution (1) appears as an objective function. As it follows from (2), its extremization results in a compromise-optimal vector of arguments. The solution of multicriteria optimization problems is described in [4, 5].

## 5. Multicriteria Evaluation Problems

In contrast to optimization problems, multicriteria evaluation is classed among analysis problems. Convolution (1) is not objective but an evaluation function, and its value quantitatively expresses the measure of quality of a multicriteria object under specified values of arguments.

Multicriteria evaluation of alternatives often needs not only analytic but also qualitative estimate. To this end, we should normalize the expression for the scalar convolution of criteria $Y(a, y_0)$ and associate the resultant value $Y_0$ with an inverted normalized fundamental scale. The general concept of a serial fundamental scale is described in [6]. Table 1 presents an interval normalized inverse scale and relates the qualitative gradations of properties of the objects and the corresponding quantitative estimates $y_0$ and $Y_0$.

**Table1**. Fundamental scale

| **Quality category** | **Ranges of normalized inverse fundamental scale for estimates** $y_0, Y_0$ |
|---|---|
| Unacceptable | 1,0 – 0,7 |
| Low | 0,7 – 0,5 |
| Satisfactory | 0,5 – 0,4 |
| Good | 0,4 – 0,2 |
| High | 0,2 – 0,0 |

The structure of the nonlinear compromise scheme allows normalizing the scalar convolution not to the maximum (usually unknown) but to the minimum value. Putting, in the expression for the nonlinear scalar convolution, the ideal (zero) values of the minimized criteria $y_{0k}(x) = 0$ and taking into account that the weight coefficients are normalized on the simplex, $\sum_{k=1}^{s} a_k = 1$, yields $Y_{0\min} = 1$. The normalized minimized scalar convolution has the form

$$Y_0 = 1 - \frac{1}{Y(a, y_0)}. \tag{3}$$

A quantitative (linguistic) estimate of an alternative can be obtained by comparing the analytic estimate $Y_0$ with the normalized inverse fundamental scale. Evaluating alternatives using the scale makes it possible to solve multicriteria problems both in traditional formulations and in the case where an alternative should be selected from a set of inhomogeneous alternatives for which a unified set of quantitative criteria cannot be formulated, and to estimate the unique alternative.

## 6. A Model Example

Let us show the capabilities of a nonlinear compromise scheme in a multicriteria analysis problem such as the quality evaluation of the glide landing of an aircraft from several criteria.

During the glide landing, the pilot navigates the aircraft using the director device. The control consists in bringing the bar crosspoint into coincidence with the central point of the device. The expression $\Delta h$ means deviation from the glide path in the vertical plane, the height $h$ is assumed zero at the instant of time $t=T$. Also, $\Delta b$ means deviation from the glide path in the horizontal plane.

At the end of the glide landing ($t=T$), the aircraft touches the runway at the point B, spaced at $\Delta l_T$ from the reference point A in the longitudinal plane and at $\Delta b_T$ from the axial line of the runway in the lateral plane.

To evaluate the landing quality, we will use three terminal ($t=T$) performance criteria $y_1$, $y_2$, $y_3$ and two integral criteria $y_4$, $y_5$:

$y_1 = |\Delta l_T| < A_1$ is the absolute value of the deviation from the reference contact point in the longitudinal plane;

$y_2 = |\Delta b_T < A_2|$ is the absolute value of the deviation of the contact point from the longitudinal axis of the runway in the lateral plane;

$y_3 = V_h^{(T)} < A_3$ is the vertical speed at the terminal point;

$y_4 = \frac{1}{T}\int_0^T |\Delta h| dt < A_4$ is the mean deviation from the glide path in the vertical plane;

$y_5 = \frac{1}{T}\int_0^T |\Delta b| dt < A_5$ is the mean deviation from the glide path in the horizontal plane.

$A_1$, $A_2$, $A_3$, $A_4$ and $A_5$ are constraints imposed on the corresponding criteria.

The scalar convolution of criteria (1) can be used as the estimate function. The weight coefficients $\alpha$ can be determined in the interactive procedure described in [4]. Let the following values of the weight coefficients be obtained:

$$\alpha_1=0.25;\ \alpha_2=0.22;\ \alpha_3=0.28;\ \alpha_4=0.16;\ \alpha_5=0.09.$$

The following values of constraints in the criteria are specified (note that the example is model):

$$A_1=15\text{ m};\ A_2=10\text{ m};\ A_3=1\text{m/sec};\ A_4=30\text{ m};\ A_5=20\text{ m}.$$

Then we will use the nonlinear compromise scheme to evaluate the quality of one landing which is characterized by the following numerical values of partial criteria:

$$y_1=6\text{ m};\ y_2=3\text{ m};\ y_3=0.2\text{ m/sec};\ y_4=10.5\text{ m and } y_5=7.25\text{ m}.$$

The normalization $y_{0k} = \frac{y_k}{A_k}, k \in [1, s]$ yields the relative partial criteria:

$$y_{01}=0.4;\ y_{02}=0.3;\ y_{03}=0.2;\ y_{04=0.35} \text{ and } y_{05}=0.36.$$

Let us calculate the scalar convolution of the criteria using the nonlinear compromise scheme (formula 1):

$$Y = 0.25\frac{1}{1-0.4} + 0.22\frac{1}{1-0.3} + 0.28\frac{1}{1-0.2} + \\ + 0.16\frac{1}{1-0.35} + 0.09\frac{1}{1-0.36} = 1.47.$$

With the normalization of (4) we get

$$Y_0 = 1 - \frac{1}{1.47} = 0.32.$$

Comparing this value with the qualitative grades of the Table 1 allows concluding that the landing was GOOD.

The multicriteria evaluation procedure is applicable for example to learning and training pilots and also in other subject domains.

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