Computer Communication & Collaboration (Vol. 2, Issue 2, 2014)
ISSN 2292-1028(Print) 2292-1036(Online)
DOIC: 2292-1036-2014-02-003-29

Asymptotic space method for multi-dimensional problems[1]

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Abstract: This paper presents a convenient method to reduce a d-dimensional problem to an arbitrary k-dimensional problem ($k < d$) for any given precision. The basic idea comes from the fact that a given compact plane region can be almost filled with a certain parameter curve. And the notion of asymptotic space is introduced from this observation. The presented method is much helpful to handle many multi-dimensional problems such as integral approximation and pattern recognition.

Keywords: Multi-dimensional problem, Asymptotic space method, Integral approximation

1. Introduction

The so-called curse of dimensionality, a term coined in [1], will arise in many multi-dimensional problems such as function approximation [2], [3], integral approximation and pattern recognition. It is willing to see that a multi-dimensional problem can be reduced to a low-dimensional one. In this paper, we build an interesting relationship between a d-dimensional problem and an arbitrary k-dimensional one ($k < d$) on the basis of the notion of asymptotic space, which can be viewed as an extension of space-filling curve [4] in some sense.

Throughout this paper, n is a positive integer in $\mathbb{Z}^+$, I is the unit interval $[0,1]$, I^d is the d-dimensional unit hypercube $[0,1]^d$.

[1] The research is funded by the National Natural Science Foundation of China, Grant No.51175425.

2. Asymptotic space

The concept of asymptotic space or asymptotic curve grew out of an intuitive phenomenon that the plane region $\Omega=[-1,1]\times[a,b]$, $(a,b\in\mathbb{R})$ can be almost filled with the curve $\cos(\mathrm{nx})$, $x\in[a,b]$ as $n\to\infty$. From this we first define the following notion of asymptotic curve:

Definition 2.1. $l(t,n)$ is called an asymptotic curve of I^2 provided for every point $p\in\mathrm{I}^2$ and $\epsilon>0$, there exist a point $p'\in A_2^1(\mathrm{I},n):=\{(x,y)\in\mathrm{I}^2 : y=l(x,n)\}$ and an integer N_ϵ, depending only on ϵ, such that $\| p-p' \|<\epsilon$ when $n>N_\epsilon$, where $\| \cdot \|$ is the Euclidean norm.

Or equivalently, $A_2^1(\mathrm{I},n)$ is dense in I^2 as $n\to\infty$. And all of the following functions on I are appropriate asymptotic curves for $A_2^1(\mathrm{I},n)$:

$$l_1(t,n)=0.5-0.5cos(n\pi t) \tag{1}$$
$$l_2(t,n)=0.5+0.5sin(n\pi t)$$
$$l_3(t,n)=0.5(1-(-1)^{[nt]})-(-1)^{[nt]+1}(nt-[nt])$$

where $[x]$ is the unique integer satisfying the inequalities $[x]\le x\le[x]+1$. In the following, any unspecified $l(t,n)$ shall be viewed as one of these three functions, the composite function $l\,l(t)$ is denoted by $l^2(t)$, and so on.

It is necessary to explain that an asymptotic space could be constructed by a function but not required (The clear examples are the Peano curve [5] the Hilbert curve [6]). Moreover, it is easy to note that an essential definition of asymptotic space should not depend on any given function at all.

Definition 2.2. $A_d^s(\mathrm{I}^s,n)$ $(s<d)$, depending only on n, is called a s-dimensional asymptotic space of I^d provided $A_d^s(\mathrm{I}^s,n)$, which is a homeomorphism of I^s, is dense in I^d as $n\to\infty$, namely, for every point $p\in\mathrm{I}^d$ and $\epsilon>0$, there exist an integer N_ϵ and a point $p'\in A_d^s(\mathrm{I}^s,n), n>N_\epsilon$ such that $\| p-p' \|<\epsilon$.

According to this definition, each sequence of points used in Monte Claro integration [7] or Quasi-Monte Claro integration [7]-[9] is an example as 0-dimensional asymptotic space of a corresponding region. Let $x=(x_1,\cdots,x_d)$, $y=(y_1,\cdots,y_d)$, $l(x,n)=(l(x_1,n),\cdots,l(x_d,n))$. According to definitions 2.1 and 2.2, we directly have

Theorem 2.1. If $l(t,n)(t\in\mathrm{I})$ is an asymptotic curve of I^2, then $A_{2d}^d(\mathrm{I}^d,n)=\{(x,y)\in\mathrm{I}^{2d} : y=l(x,n)\}$ is a d-dimensional asymptotic space of I^{2d}.

Remark 2.1. This conclusion can also be stated as follows: If $A_2^1(\mathrm{I},n)$ is an asymptotic space of I^2, then the finite direct sum $A_{2d}^d(\mathrm{I}^d,n)=A_2^1(\mathrm{I},n)\oplus\cdots\oplus A_2^1(\mathrm{I},n)$ is a d-dimensional asymptotic space of I^{2d}.

A natural question is that, for any dimension $4d$, can we get a d-dimensional asymptotic space of I^{4d} by repetition of the procedure above? The answer is no and here is a counter example:

Example 2.1. Suppose $A_4^2=(x_1,x_2,l(x_1),l(x_2))$ is a 2-dimensional asymptotic space of I^4 and $A_2^1=(x_1,l(x_1))$ is a 1-dimensional asymptotic curve of I^2. Let

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$$A_4^1 = A_4^2 A_2^1 = (x_1, x_2, l(x_1), l(x_2)) \square (x_1, l(x_1))$$
$$= (x_1, l(x_1), l(x_1), l^2(x_1))$$

Take $x = (a,1,0,b) \in \mathrm{I}^4$, for any $x' = (t, l(t), l(t), l^2(t)) \in A_4^1$ $(t \in \mathrm{I})$

$$\| x' - x \| \geq \sqrt{(l(t) - 1)^2 + (l(t) - 0)^2}$$
$$= \sqrt{2[l(t) - 1/2]^2 + 1/2} \geq \sqrt{2}/2$$

So A_4^1 is not a 1-dimensional asymptotic space of I^4. As the case stands, $A_4^1 = (x_1, l(x_1), l(x_1), l^2(x_1))$ is just an asymptotic space of I^3 with a 45-degree counterclockwise rotation about the x_1-axis.

The following theorem provides an easy approach to construct an asymptotic space.

Theorem 2.2. If $l(t,n)(t \in \mathrm{I})$ is an asymptotic curve of I^2, then $A_3^1(\mathrm{I}, n) = \{(x,y,z) \in \mathrm{I}^3 : y = l(x), z = l^2(x)\}$ is a 1-dimensional asymptotic space of I^3.

Proof. Notice that $l(t,n)$ is a mapping of $[(i-1)/n, i/n]$ onto $\mathrm{I}(0 \leq i \leq n)$, then the composite function $l^2(t,n) = l(l(t,n),n)$ is a mapping of $[(i-1)(j-1)/n^2, ij/n^2]$ onto I. Hence, for any point $p = (x,y,z) \in \mathrm{I}^3$, there exist i, j and $x' \in \mathrm{I}$ such that $|x' - x| \leq 1/n$, $|l(x',n) - y| \leq 1/n$, and $l^2(x',n) = z$, namely, there exists a point $p' = (x', l(x',n), l^2(x',n)) \in A_3^1(\mathrm{I}, n)$ such that $\| p - p' \| \leq \sqrt{2}/n$. □

The following corollary is immediate.

Corollary 2.1. If $l(t,n)(t \in \mathrm{I})$ is an asymptotic curve of I^2, then

$$A_d^s(\mathrm{I}^s, n) = \{x \in \mathrm{I}^d : x' \in \mathrm{I}^s, x_{s+k} = l^k(x_s)\}, 1 \leq k \leq d - s$$

is an s-dimensional asymptotic space of I^d, where $x' = (x_1, \cdots, x_s)$.

As a variation, the following corollary is easily obtained from Theorem 2.1 and corollary 2.1.

Corollary 2.2. If $A_{d_i}^{s_i}(\mathrm{I}^{s_i}, n)$ are s_i-dimensional asymptotic space of I^{d_i} $(1 \leq i \leq m)$, then the direct sum

$$A_{d_1+\cdots+d_m}^{s_1+\cdots+s_m}(\mathrm{I}^{s_1+\cdots+s_m}, n) = A_{d_1}^{s_1}(\mathrm{I}^{s_1}, n) \oplus \cdots \oplus A_{d_m}^{s_m}(\mathrm{I}^{s_m}, n)$$

is a $(s_1 + \cdots + s_m)$-dimensional asymptotic space of $\mathrm{I}^{d_1+\cdots+d_m}$.

3. A kind of approximate relationship

We first consider the integrability of a function over an asymptotic space $A_d^s(\mathrm{I}^s, n)$.

Lemma 3.1. Let $A_d^s(\mathrm{I}^s, n)$ be an asymptotic space of I^d. If $f(x)$ is a continuous integrable function over I^d, then the restriction $h(x) = f(x) | A_d^s(\mathrm{I}^s, n)$ is still a continuous integrable function over $A_d^s(\mathrm{I}^s, n)$.

Proof. Let $x, x' \in A_d^s(\mathrm{I}^s, n)$, then $x, x' \in \mathrm{I}^d$ since $A_d^s(\mathrm{I}^s, n) \subset \mathrm{I}^d$. As $f(x)$ is continuous, so we have, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|h(x) - h(x')| = |f(x) - f(x')| < \epsilon, \text{ whenever } \| x - x' \| < \delta$$

So $h(x)$ is continuous on $A_d^s(\mathrm{I}^s, n)$.

For a given $n \in \mathbb{Z}^+$, the Lebesgue measure of asymptotic space $A_d^s(\mathrm{I}^s, n)$ is finite; and $g(x)$ is a bounded measurable function according to the integrability of $h(x)$. So $g(x)$ is integrable over $A_d^s(\mathrm{I}^s, n)$ in the sense of Lebesgue, and it was already known that $g(x)$ is continuous on $A_d^s(\mathrm{I}^s, n)$, hence, $g(x)$ is Riemann integrable over $A_d^s(\mathrm{I}^s, n)$. □

Given an asymptotic curve $l(t, n)$, for any $x \in \mathrm{I}$, the pair $(x, l(x, n))$ is a point of $A_2^1(\mathrm{I}, n)$. The following theorem claims that a 2-dimensional integral over I^2 can be viewed as one over $A_2^1(\mathrm{I}, n)$ as $n \to \infty$. This is essentially the approximate relationship between a 2-dimensional integral and a 1-dimensional one.

Theorem 3.1. Let $l(t, n)$ be an asymptotic curve of I^2 and $f(x, y)$ be a continuous integrable function over I^2, then we have

$$\int_0^1 \int_0^1 f(x, y)\mathrm{d}x\mathrm{d}y = \lim_{n\to\infty} \int_0^1 f(x, l(x, n))\mathrm{d}x \tag{2}$$

Proof. Let

$$H(x) = \int_0^1 f(x, y)\mathrm{d}y \tag{3}$$

Then

$$\int_0^1 \int_0^1 f(x, y)\mathrm{d}x\mathrm{d}y = \int_0^1 H(x)\mathrm{d}x \tag{4}$$

According to the definition of Riemann integral

$$\int_0^1 H(x)\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(x'), \ \forall x' \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \tag{5}$$

On each interval $[(i-1)/n, i/n]$ $(i = 1, 2, \cdots, n)$, $l(x, n)$ is monotonic and x' is arbitrary, so making

$$x' = \frac{i-1}{n} + l^{-1}(y, n), \ \forall y \in [0,1] \tag{6}$$

Therefore, the right-hand side of (5) has the expression

ISSN 2292-1028(Print) 2292-1036(Online)
DOIC: 2292-1036-2014-02-003-29

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\int_0^1 \frac{1}{n} f\left(\frac{i-1}{n}+l^{-1}(y,n),y\right)\mathrm{d}y \\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\int_0^1\int_{\frac{i-1}{n}}^{\frac{i}{n}} f\left(\frac{i-1}{n}+l^{-1}(y,n),y\right)\mathrm{d}x\mathrm{d}y \\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\int_0^1\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x,l(x,n))\mathrm{d}x\mathrm{d}y \\
&=\lim_{n\to\infty}\int_0^1\int_0^1 f(x,l(x,n))\mathrm{d}x\mathrm{d}y \\
&=\lim_{n\to\infty}\int_0^1 f(x,l(x,n))\mathrm{d}x
\end{aligned}$$

That is, (2) holds. □

Further, observe that the 2-dimensional integral of (2) can be easily extended to the 2*d*-dimensional one, let $x=(x_1,\cdots,x_d)$, $y=(y_1,\cdots,y_d)$, $l(x,n)=(l(x_1,n),\cdots,l(x_d,n))$, the following conclusion can be obtained directly from Theorem 3.1.

Theorem 3.2. Let $l(t,n)$ be an asymptotic curve of I^2 and $f(x,y)$ be a continuous integrable function over I^{2d}, then we have

$$\int_{\mathrm{I}^{2d}} f(x,y)\mathrm{d}x\mathrm{d}y=\lim_{n\to\infty}\int_{\mathrm{I}^d} f(x,l(x,n))\mathrm{d}x \tag{7}$$

By repeating the proof process of Theorem 3.1 for *d*-dimensional integral over I^d, we obtain the following extension of Theorem 3.1.

Theorem 3.3. Let $f(x)$ be a continuous integrable function over I^d, $A_d^s(\mathrm{I}^s,n)$ be an *s*-dimensional asymptotic space of I^d, and $h(t,n)=f(x)\,|\,A_d^s(\mathrm{I}^s,n), t\in\mathrm{I}^s$ be the restriction of $f(x)$ to $A_d^s(\mathrm{I}^s,n)$, then we have

$$\int_{\mathrm{I}^d} f(x)\mathrm{d}x=\lim_{n\to\infty}\int_{\mathrm{I}^s} h(t,n)\mathrm{d}t \tag{8}$$

Remark 3.1. If our approach to the construction of an *s*-dimensional asymptotic space of I^d is limited to Corollary 2.2, then for a given asymptotic curve $l(t,n)$, there are only *s* kinds of definition of $h(t,n)$. That is , $h(t,n)=f\left(t_1,\dots,t_s,l(t_k,n),\dots,l^{d-s}(t_k,n)\right)1\le k\le s.$

Remark 3.2. This relationship was used to evaluate the small probability integral risen in the uncertainty importance analysis [10].

From Corollary 2.2 and Theorem 3.1, we can have the further conclusion.

Theorem 3.4. Let $A_{d_i}^{s_i}(\mathrm{I}^{s_i},n)$ be s_i-dimensional asymptotic space of I^{d_i} $(s_i<d_i,\ 1\le i\le m)$, $f(x)$ be a continuous integrable function over $\mathrm{I}^{d_1+\cdots+d_m}$, and $h(t,n), t\in\mathrm{I}^{s_1+\cdots+s_m}$ be the restriction of $f(x)$ to the direct sum space

$$A_{d_1+\cdots+d_m}^{s_1+\cdots+s_m}(\mathrm{I}^{s_1+\cdots+s_m},n)=A_{d_1}^{s_1}(\mathrm{I}^{s_1},n)\oplus\cdots\oplus A_{d_m}^{s_m}(\mathrm{I}^{s_m},n)$$

then we have

$$\int_{\mathrm{I}^{d_1+\cdots+d_m}} f(x)\mathrm{d}x = \lim_{n\to\infty}\int_{\mathrm{I}^{s_1+\cdots+s_m}} h(t,n)\mathrm{d}t \tag{9}$$

4. The dependence of error behavior on *n*

The dependence of error behavior on n is discussed in this section.

Theorem 4.1. Let $l(x,n)$ be an asymptotic curve of I^2, $f(x,y)$ be a continuous integrable function over I^2, then we have

$$\left|\int_0^1\int_0^1 f(x,y)dxdy - \int_0^1 f(x,l(x,n))dx\right| \le \omega(1/n) \tag{10}$$

Where

$$\omega(\delta) = \sup_{|x'-x|<\delta}\left|f(x',y)-f(x,y)\right|, \forall y\in \mathrm{I}, \delta>0 \tag{11}$$

Proof. Obviously, the second integral on the left-hand side of (10)

$$\int_0^1 f\left(x,l(x,n)\right)\mathrm{d}x = \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x,l(x,n))\mathrm{d}x \tag{12}$$

As $l(x,n)$ is monotonic on each interval $\left[\left(\mathrm{i}-1\right)/\mathrm{n},\mathrm{i}/\mathrm{n}\right]$ $(i=1,2,\cdots,n)$, so we have, for an odd i

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x,l(x,n))\mathrm{d}x = \frac{1}{n}\int_0^1 f\left(\frac{i-1}{n}+l^{-1}(y,n),y\right)\mathrm{d}y \tag{13}$$

and for an even i

$$\begin{aligned}\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x,l(x,n))\mathrm{d}x &= -\frac{1}{n}\int_1^0 f\left(\frac{i-1}{n}+l^{-1}(y,n),y\right)\mathrm{d}y \\ &= \frac{1}{n}\int_0^1 f\left(\frac{i-1}{n}+l^{-1}(y,n),y\right)\mathrm{d}y\end{aligned} \tag{14}$$

It follows from (12), (13) and (14) that

$$\int_0^1 f(x,l(x,n))\mathrm{d}x = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 f\left(\frac{i-1}{n}+l^{-1}(y,n),y\right)\mathrm{d}y \tag{15}$$

Similarly, the first integral on the left-hand side of (10)

$$\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{n}\int_0^1\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x,y)\mathrm{d}x\mathrm{d}y \tag{16}$$

According to the second Mean Value Theorem for integrals, there exists a point x' in each interval $[(i-1)/n, i/n]$ $(i=1,2,\cdots,n)$ such that

$$\int_0^1\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{n}\int_0^1 f(x',y)\mathrm{d}y \tag{17}$$

It follows from (16) and (17) that

$$\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 f(x',y)\mathrm{d}y \tag{18}$$

We infer from (15) and (18) that

$$\left|\int_0^1\int_0^1 f(x,y)dxdy - \int_0^1 f(x,l(x,n))dx\right|$$
$$= \left|\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left[f(x',y) - f\left(\frac{i-1}{n} + l^{-1}(y,n),y\right)\right]dy\right|$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left|\left[f(x',y) - f\left(\frac{i-1}{n} + l^{-1}(y,n),y\right)\right]\right|dy$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \omega\left(\frac{1}{n}\right)dy = \omega\left(\frac{1}{n}\right) \tag{19}$$

□

Theorem 4.2. Under the conditions of Theorem 4.1, if $\frac{\partial f}{\partial x}(x,y)$ is continuous and bounded on I^2, then we have

$$\left|\int_0^1\int_0^1 f(x,y)dxdy - \int_0^1 f(x,l(x,n))dx\right| \leq \frac{C}{n} \tag{20}$$

Where

$$C = \sup_{\forall x,y\in \mathrm{I}}\left|\frac{\partial f}{\partial x}(x,y)\right| \tag{21}$$

Proof. According to Eq.(19), notice that $f(x',y) = f(x'',y) + \frac{\partial f}{\partial x}(x'',y)(x'-x'')$, where $x' \in [(i-1)/n, i/n]$ and $x'' = \frac{i-1}{n} + l^{-1}(y,n)$, then we have

$$|f(x',y) - f(x'',y)| = \left|\frac{\partial f}{\partial x}(x'',y)(x'-x'')\right| \leq \frac{C}{n} \tag{22}$$

So the conclusion holds from Eqs.(19) and (22). □

Theorem 4.2. Under the conditions of Theorem 3.4, if $\frac{\partial f}{\partial x_i}(x), i = 1,\cdots,(d_1+\cdots+d_m)$ are continuous and bounded on $\mathrm{I}^{d_1+\cdots+d_m}$, then we have

$$\left|\int_{\mathrm{I}^{d_1+\cdots+d_m}} f(x)dx - \int_{\mathrm{I}^{s_1+\cdots+s_m}} h(t,n)dt\right| \leq \frac{C}{n}\sum_k (d_k - s_k) \tag{23}$$

Where

$$C = \max_i \sup_{\forall x,y\in \mathrm{I}}\left|\frac{\partial f}{\partial x_i}(x)\right| \tag{24}$$

5. An expansion of a function on asymptotic space

For any given function $f \in C(I^d)$, we have an orthogonal expansion of order s

$$f(x) = \sum_{i=0}^{s} \alpha_i \phi_i(x)$$

where $\{\phi_i\}_i$ is an orthogonal bases on I^d and

$$\alpha_i = \int_{I^d} f(x)\phi_i(x)dx,\ 0 \le i \le s$$

Suppose $A_d^k(I^d, n)$ is a k -dimensional asymptotic space of I^d, then we have

$$\alpha_i = \lim_{n\to\infty} \int_{I^k} (f(x)\phi_i(x))\big|A_d^k(I^d,n)dx,\ 0 \le i \le s$$

For a fixed n, we have

$$\hat{\alpha}_i = \int_{I^k} (f(x)\phi_i(x))\big|A_d^k(I^d,n)dx,\ 0 \le i \le s$$

then we have an estimation of f

$$\hat{f}(x) = \sum_{i=0}^{s} \hat{\alpha}_i \phi_i(x)$$

This is usually referred to as the expansion of a function on asymptotic space.

6. Discussion

By introducing the notion of asymptotic space, a kind of approximate relationship between a multi-dimensional problem and a low-dimensional one is built in this paper. And we also discussed the dependence of error behavior on the asymptotic parameter n.

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