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Statistical analysis of a new correlation peak detection method for unimodal autocorrelation

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Abstract: A new, so-called seeking-and-tracking correlators for peak detection is proposed and its impact on synchronization used by the receiver of a binary chirp modulation system is analyzed. In this paper, the probability of the zero delay correlation peak of the correlation function of chirp modulated signals is calculated assuming an AWGN channel in four cases: using the legacy sliding and the new seeking-and-tracking correlators with coherent and non-coherent receiving, respectively. During the calculation of correlation using the seeking-and-tracking correlators we suppose that the noise components are not statistically independent, thus this effect enables to reduce the error probability of correlation peak detection.

The results can be used for spectrum spread modulation based systems, where the autocorrelation function of the spectrum spreading code has only one well-determined peak, e.g. DS (Direct Sequence) and FFH (Fast Frequency Hopping). Such systems are unimodal, the code and symbol synchronization are either before or after the carrier synchronization, i.e. the phase of the carrier still unknown or just known during the establishing period of the code and symbol synchronization.

Keywords: Synchronization, Spectrum Spread Modulation Based System, WLAN-positioning, AWGN Channel, Error Probability

1 Introduction

In the last decade new wireless technologies have tried to tackle the exponential increasing demand for higher data rates and higher mobility [1]. Some technologies, e.g. 3 and 4G, WLAN (Wireless Local Area Network) have been evolved using different sophisticated techniques e.g. MIMO (Multiple Input Multiple Output) and new coding procedures, like turbo codes, LDPC (Low-Density Parity-Check) codes to boost up existing capacities of wireless links, and some technologies just emerge e.g. UWB and cognitive radios to share and reuse the radio spectrum, therefore optimize its utilization.

For proper operation, the receiver has to synchronize with the incoming signal. This means timing synchronization, when the receiver needs to determine at which time instants the incoming signal has to be sampled, and carrier synchronization, when the receiver needs to adopt the frequency and phase of its local carrier oscillator with those of the received signal, too. The accuracy of synchronization has a major effect on the performance of communication independently from the technology. However, if an unfriendly, but realistic wireless environment is assumed with low SNR (Signal-to-Noise Ratio), fading, multipath propagation, in-system and out-system interference, then it is easy to see, that the acquisition of the synchronization parameters is difficult. Therefore synchronization should be considered in general as a challenging task.

In this paper, a simple synchronization scheme using two correlators are proposed and its statistical behavior on correlation peak detection using binary chirp modulation via an AWGN channel is analyzed. The so-called seeking-and-tracking correlators enable to acquire overlapping noise components during the measurements, which means that these components are not statistically independent. The advantage of this property during the calculation of correlation is taken into account, hence the error probability of correlation peak detection is reduced. The proposed procedure ensures higher accuracy for synchronization than using the legacy sliding correlator. The findings of this paper can be adopted for spectrum spread modulation based communication systems, where the autocorrelation function of the spectrum spreading code has only one well-determined peak, e.g. DS and FFH based technologies.

Other possible application areas are the UWB (Ultra-wideband) and WLAN-based localization systems. There are a couples of techniques for localization [2], e.g. AoA (Angle of Arrival) [3]-[7], ToA (Time of Arrival) [8]-[13] and TDOA (Time Difference of Arrival) [14]-[16], which require precise synchronization. Therefore the accuracy of positioning are closely related to the accuracy of synchronization.

The paper is organized as follows. Section II introduces the system model, i.e. the basics of a chirp modulated system. Section III and Section IV describe the synchronization systems and the joint probability density functions of the two correlator scenarios in non-coherent and coherent cases, respectively. Section V presents the numerical results of our calculations. Finally, Section VI concludes the paper.

2 System model

The basic signals assigned to each bit in the applied binary chirp system are $g_{c1}(t)$, $g_{c2}(t)$ and the holders of them are $[-T_c/2,\ T_c/2)$. The two low-pass equivalent complex-valued representations of the basic signals are as follows:

$$\begin{aligned}
&\text{for } 0 \quad g_{c1}(t) = A\exp\left(j2p\,\frac{\Delta f}{2T_c}t^2 \right), \quad t \in \left[-\frac{T_c}{2}, \frac{T_c}{2} \right) \\
&\text{for } 1 \quad g_{c2}(t) = A\exp\left(-j2p\,\frac{\Delta f}{2T_c}t^2 \right), \quad t \in \left[-\frac{T_c}{2}, \frac{T_c}{2} \right)
\end{aligned} \tag{1}$$

where T_c is the basic signals' symbol time, Δf is the chirp modulated signal frequency spreading domain and A is the amplitude of the frequency changing sinus signal.

The instantaneous frequency changing of the signal is presented on Figure 1. The unit of the time domain is 10^{-5} and the unit of the frequency domain is 10^{5} accordingly.

3 Statistical analysis of the non-coherent system

In this section the non-coherent system model, the joint probability density function (PDF) of the correlators' output signals and the decision error probability for both seeking-and-tracking and sliding correlators are presented.

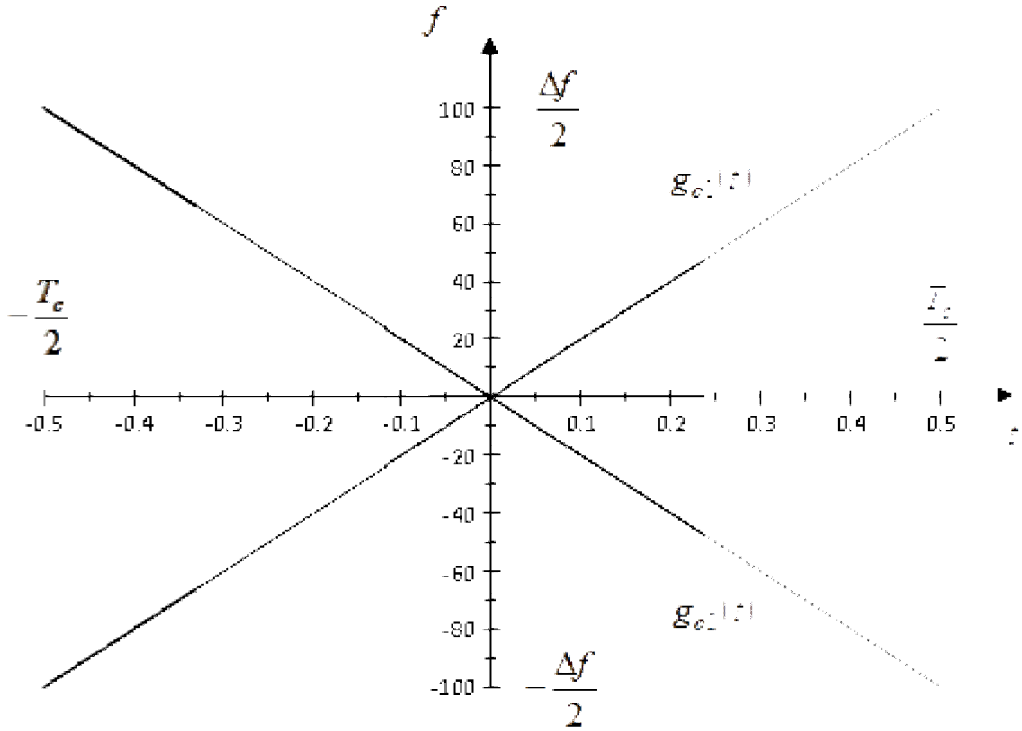


Figure 1 Instantaneous frequency of the chirp signal in function of time, when $T_c = 1, \Delta f = 200$

3.1 Non-coherent synchronization system using training sequence

The non-coherent synchronization system using training sequence is shown in Figure 2. The goal of the receiver is the detection of the correlation function maximum at $\tau = 0$ position. This means non-coherent measuring of the correlation function's values belonging to 0 and τ delays and deciding on the greater of them. It is assumed that the transmitter sends a training sequence, e.g. $g_{c1}(t)$to the receiver, while the synchronization is not finished. Note that the synchronization is possible without training sequence. The statistical properties of the given system by Figure 2 are analyzed in two scenarios.

In the first scenario it is supposed that 0 and τ delays of $z_1(0)$ and $z_1(\tau)$ values are generated in overlapping time windows, i.e. two correlators are used, a seeking and a tracking correlator at the same time. This is illustrated by Figure 3a. The integration domain of the correlators are normalized to $[-T_c/2, T_c/2)$. It is supposed that the incoming signal of one correlator is received in appropriate synchronposition and the other correlator gets it with τ delay due to practical reasons. In this scenario the two kinds of correlation calculation are evaluated in overlapping time windows, where the AWGN noises are not independent from each other, i.e. the results of the correlation are dependent random variables (RV).

In the second scenario it is assumed that the measurements of 0 and τ delays are performed in non-overlapping time windows, i.e. only one sliding correlator is used for the detection of the maximum position. Figure 3b shows this case. The integration domain of the correlators are normalized to $[-T_c/2, T_c/2)$ again. It is supposed that the incoming signal of one correlator is received in appropriate synchronposition, but the other one receives it with $\tau\text{-}nT_c$ delay, i.e. the two kinds of correlation calculation is not evaluated in overlapping time windows, thus the results of the correlation are treated as independent RVs.

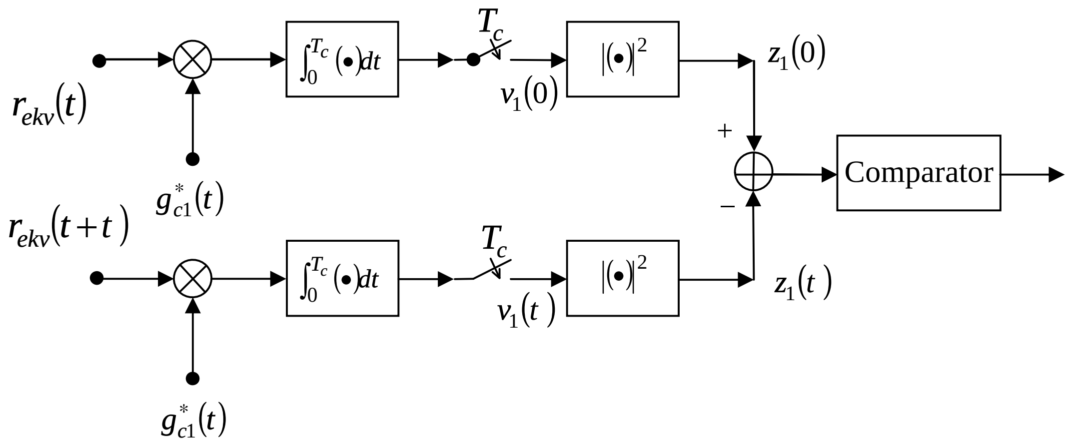


Figure 2 Non-coherent synchronization system using training sequence

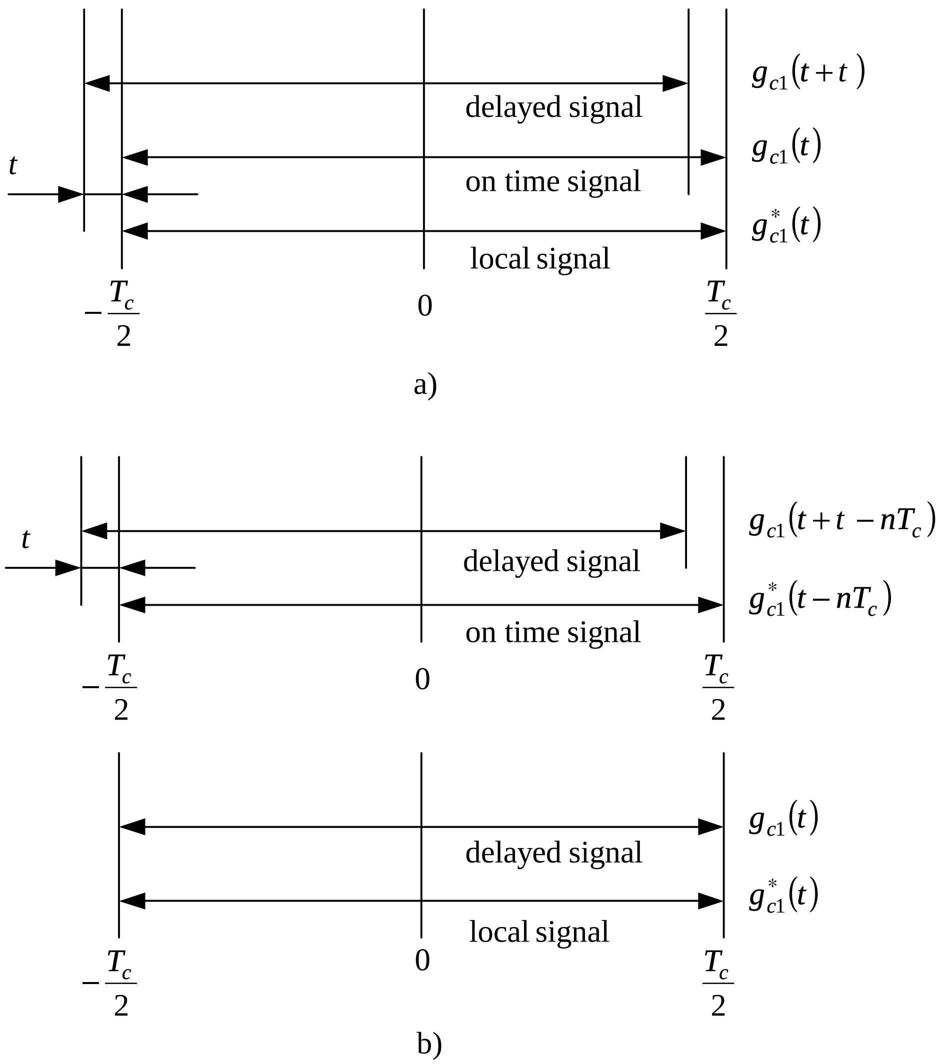


Figure 3 The integration domains of the different scenarios
a) Seeking-and-tracking correlators, b) Sliding correlator

The synchronposition is determined by choosing the greater signal from the result of the two correlation calculation. The null comparator is used for this purpose, i.e. comparing the correlators' output signals and choose the bigger one as shown in Figure 2.

The operation of the synchronizing system is given as follows:

$$z_1(t) = |v_1(t)|^2 = \left| \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} r_{ekv}(t+t) g_{c1}^*(t) dt \right|^2 = \left| \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} r_{ekv}(t+t) A \exp\left(-j2p \frac{\Delta f}{2T_c} t^2 \right) dt \right|^2 , \quad (2)$$

where

$$r_{ekv}(t+t) = A g_1(t+t) + A g_1(t - T_c + t) + n(t+t), \quad (3)$$

and $n(t) = n_1(t) + jn_2(t)$ is an independent complex Gaussian distributed stochastic process with zero expected value and N_0 power density per complex dimension. The value of $z_1(\tau)$ can be expressed after substitutions with:

$$z_1(t) = |v_1(t)|^2 = \left| \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} r_{ekv}(t) g_{c1}^*(t) dt \right|^2 =$$
$$\left| A^2 \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t} g_{c1}(t+t) g_{c1}^*(t) dt + A^2 \int_{\frac{T_c}{2}-t}^{\frac{T_c}{2}} g_{c1}(t - T_c + t) g_{c1}^*(t) dt + A \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+t) g_{c1}^*(t) dt \right|^2 , \quad (4)$$

where $v_1(\tau)$ is the complex output signal of the integrator at the end of the integration time (see Figure 2).

As a reminder, the task is to determine the probability of $z_1(0) < z_1(\tau)$ with given signal-to-noise ratio ($\gamma = E_b/N_0$), i.e. according to the appropriate synchronposition the τ value is chosen based on the given measurement. To simplify the calculation, the second part of (4) is neglected because of the properties of the chirp modulation. So one can get the following expression:

$$z_1(t) = |v_1(t)|^2 \cong \left| A^2 \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t} g_{c1}(t+t) g_{c1}^*(t) dt + A \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+t) g_{c1}^*(t) dt \right|^2 =$$
$$\left| A^2 \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t} \exp\left(j2p \frac{\Delta f}{2T_c} (t+t)^2 \right) \exp\left(-j2p \frac{\Delta f}{2T_c} t^2 \right) dt + A \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+t) \exp\left(-j2p \frac{\Delta f}{2T_c} t^2 \right) dt \right|^2 . \quad (5)$$

The expected value of non-coherent received complex signal and the stochastic part of the complex Gaussian distributed RV can be derived from the first and second part of (5), respectively.

The next task is the calculation of the aforementioned expected value as follows:

$$\mathrm{E}\{v_1(t)\} = A^2 \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t} \exp\left(j2p \frac{\Delta f}{2T_c}(t+t)^2 \right) \exp\left(-j2p \frac{\Delta f}{2T_c} t^2 \right) dt =$$

$$A^2 \exp\left(j2p \frac{\Delta f}{2T_c} t^2 \right) \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t} \exp\left(j2p \frac{\Delta f}{T_c} tt \right) dt =$$

$$= A^2 \exp\left(j2p \frac{\Delta f}{2T_c} t^2 \right) \left[\frac{\exp\left(j2p \frac{\Delta f}{T_c} tt \right)}{j2p \frac{\Delta f}{T_c} t} \right]_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t} =$$

$$A^2 \exp\left(j2p \frac{\Delta f}{2T_c} t^2 \right) \frac{\exp\left(j2p \frac{\Delta f}{T_c} \left(\frac{T_c}{2} - t \right) t \right) - \exp\left(-j2p \frac{\Delta f}{T_c} \frac{T_c}{2} t \right)}{j2p \frac{\Delta f}{T_c} t} =$$

$$A^2 \frac{\exp\left(jp\Delta f t \left(\frac{T_c - t}{T_c} \right) \right) - \exp\left(-jp\Delta f t \left(\frac{T_c - t}{T_c} \right) \right)}{j2p \frac{\Delta f}{T_c} t} = \qquad (6)$$

$$2E_b \left(1 - \frac{t}{T_c} \right) \frac{\sin\left(p\Delta f t \left(1 - \frac{t}{T_c} \right) \right)}{p\Delta f t \left(1 - \frac{t}{T_c} \right)},$$

where $E_b = T_b A^2/2$. This formula is an even function according to τ, thus the expected value of the signal when the delay's sign is arbitrary can be determined by the following expression:

$$\mathrm{E}\{v_1(t)\} = 2E_b \left(1 - \frac{|t|}{T_c} \right) \frac{\sin\left(p\Delta f |t| \left(1 - \frac{|t|}{T_c} \right) \right)}{p\Delta f |t| \left(1 - \frac{|t|}{T_c} \right)}. \qquad (7)$$

The second part of (5) is a complex Gaussian distributed RV with zero expected value. Its deviation is given as follows:

$$
\begin{aligned}
&\mathrm{E}\left\{A^2\left|\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+t)g_{c1}^*(t)dt\right|^2\right\} = \\
&\mathrm{E}\left\{A^2\left(\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+t)g_{c1}^*(t)dt\right)\left(\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n^*(s+t)g_{c1}(s)ds\right)\right\} = \\
&A^2\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} \mathrm{E}\{n(t+t)n^*(s+t)\}g_{c1}^*(t)g_{c1}(s)dtds = \\
&A^2\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} 2N_0 d(t-s)g_{c1}^*(t)g_{c1}(s)dtds = A^2 2N_0\int_{\frac{T_c}{2}}^{\frac{T_c}{2}}|g_{c1}(t)|^2 dt = 2E_b 2N_0.
\end{aligned}
\tag{8}
$$

From this, the output signal of the two correlators in Figure 2 can be expressed by

$$
\begin{aligned}
z_1(0) &= \left|2E_b + \sqrt{2E_b}n_0\right|^2 = \left|2E_b + \sqrt{2E_b}n_{01} + j\sqrt{2E_b}n_{02}\right|^2, \\
z_1(t) &= \left|2E_b r + \sqrt{2E_b}n_t\right|^2 = \left|2E_b r + \sqrt{2E_b}n_{t1} + j\sqrt{2E_b}n_{t2}\right|^2,
\end{aligned}
\tag{9}
$$

where n_0 and n_τ are complex Gaussian distributed random variables with $2N_0$ deviation, n_{01}, n_{02}, $n_{\tau 1}$and$n_{\tau 2}$ are of these pair-wise independent real and imaginary parts respectively with N_0 deviation. The correlation parameter between n_{01}, $n_{\tau 1}$ and between n_{02}, $n_{\tau 2}$ is the following:

$$
r = \left(1-\frac{|t|}{T_c}\right)\frac{\sin\left(p\Delta f|t|\left(1-\frac{|t|}{T_c}\right)\right)}{p\Delta f|t|\left(1-\frac{|t|}{T_c}\right)}. \tag{10}
$$

The correlation for n_{01} and $n_{\tau 1}$ can be calculated as follows:

$$\frac{\mathrm{E}\left[n_{01}n_{t1}^{*}\right]}{\sqrt{\mathrm{E}\left[n_{01}^{2}n_{t1}^{2}\right]}}=\frac{\mathrm{E}\left[\left(\frac{1}{\sqrt{T_c}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}n_1(t)g_{c1}^{*}(t)dt\right)\left(\frac{1}{\sqrt{T_c}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}n_1^{*}(t+t)g_{c1}(t)dt\right)\right]}{N_0}$$

$$=\frac{\mathrm{E}\left[\frac{1}{T_c}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}n_1(t)n_1^{*}(s+t)g_{c1}^{*}(t)g_{c1}(s)\,ds\,dt\right]}{N_0}$$

$$=\frac{\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}\mathrm{E}\left[n_1(t)n_1^{*}(s+t)\right]g_{c1}^{*}(t)g_{c1}(s)\,ds\,dt}{N_0T_c}=\frac{1}{T_c}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}}d(t-s-t)g_{c1}^{*}(t)g_{c1}(s)\,ds\,dt$$

$$=\frac{1}{T_c}\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-t}g_{c1}^{*}(s+t)g_{c1}(s)\,ds=\left(1-\frac{|t|}{T_c}\right)\frac{\sin\left(p\Delta ft\left(1-\frac{|t|}{T_c}\right)\right)}{p\Delta f|t|\left(1-\frac{|t|}{T_c}\right)}=r. \tag{11}$$

The correlation function of the chirp modulated signal (for low τ/T_c) is illustrated in Figure 4. It is clearly visible that the correlation quickly decreases according to the increase of $|\tau|$.

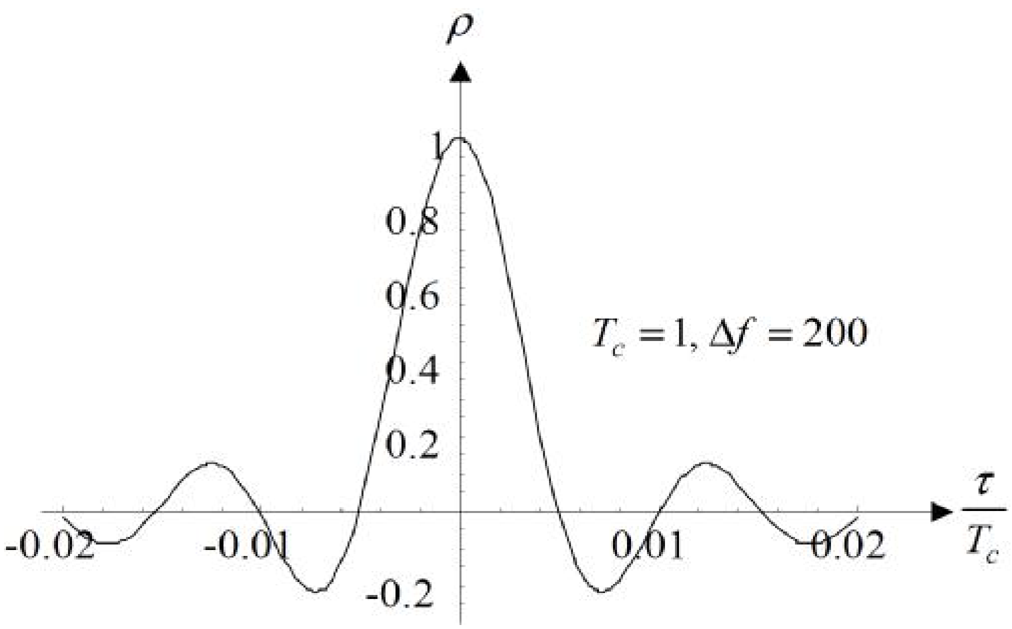


Figure 4 The correlation function of the chirp modulated signal in function of the delay's relative value

3.2 Joint probability density function of the correlators' output signals and the probability of false decision using non-coherent seeking-and-tracking correlators

Now, the joint probability density function of the output signals of (8) and (9) are going to be determined. Let

$$
\begin{gathered}
\Re(v_1(0)) = X = 2E_b + \sqrt{2E_b}n_{01},\ \mathrm{E}(X) = 2E_b,\ \mathrm{E}\left(\left(X - \mathrm{E}(X)^2\right)\right) = 2E_b N_0 = s^{\ 2}, \\
\Im(v_1(0)) = Y = \sqrt{2E_b}n_{02},\ \mathrm{E}(Y) = 0,\ \mathrm{E}\left(\left(Y - \mathrm{E}(Y)^2\right)\right) = 2E_b N_0 = s^{\ 2}, \\
\Re(v_1(t)) = V = 2E_b r + \sqrt{2E_b}n_{t1},\ \mathrm{E}(V) = 2E_b r,\ \mathrm{E}\left(\left(V - \mathrm{E}(V)^2\right)\right) = 2E_b N_0 = s^{\ 2}, \\
\Im(v_1(t)) = Z = \sqrt{2E_b}n_{t2},\ \mathrm{E}(Z) = 0,\ \mathrm{E}\left(\left(Z - \mathrm{E}(Z)^2\right)\right) = 2E_b N_0 = s^{\ 2}, \\
\mathrm{E}(XY) = 0,\ \mathrm{E}(VZ) = 0, \\
\mathrm{E}((X - \mathrm{E}(X))(V - \mathrm{E}(V))) = 2E_b N_0 r,\ \mathrm{E}((Y - \mathrm{E}(Y))(Z - \mathrm{E}(Z))) = 2E_b N_0 r.
\end{gathered}
\qquad (12)
$$

The joint probability density function can be defined as follows (*T_c = 1 and $2E_b$ = 1)*:

$$
\begin{aligned}
& f_{XYVZ}(x, y, v, z) = \left(\frac{1}{2ps^{\ 2}(1 - r^{\ 2})}\right)^2 \\
& \exp\left(-\frac{1}{2s^{\ 2}(1 - r^{\ 2})}\left((x-1)^2 - 2r(x-1)(v - r) + (v - r)^2\right)\right) \\
& \exp\left(-\frac{1}{2s^{\ 2}(1 - r^{\ 2})}\left(y^2 - 2ryz + z^2\right)\right).
\end{aligned}
\qquad (13)
$$

With the following transformations

$$
\begin{aligned}
& R_1 = \sqrt{X^2 + Y^2},\ \Phi_1 = \mathrm{artg}\left(\frac{Y}{X}\right), \qquad X = R_1\cos(\Phi_1),\ Y = R_1\sin(\Phi_1), \\
& R_2 = \sqrt{V^2 + Z^2},\ \Phi_1 = \mathrm{artg}\left(\frac{Z}{V}\right), \qquad V = R_2\cos(\Phi_2),\ Z = R_2\sin(\Phi_2),
\end{aligned}
\qquad (14)
$$

one can get:

$$
\begin{aligned}
& f_{R_1R_2\Phi_1\Phi_2}(r_1, r_2, j_1, j_2) = r_1, r_2\left(\frac{1}{2ps^{\ 2}(1 - r^{\ 2})}\right)^2 \\
& \exp\left(-\frac{1}{2s^{\ 2}(1 - r^{\ 2})}\left((r_1\cos j_1 - 1)^2 - 2r(r_1\cos j_1 - 1)(r_2\cos j_2 - r) + (r_2\cos j_2 - r)^2\right)\right) \\
& \exp\left(-\frac{1}{2s^{\ 2}(1 - r^{\ 2})}\left((r_1\sin j_1)^2 - 2r(r_1\sin j_1)(r_2\sin j_2) + (r_2\sin j_2)^2\right)\right) =
\end{aligned}
$$

$$= r_1, r_2 \left(\frac{1}{2p s^2 (1 - r^2)} \right)^2 \exp\left(-\frac{1}{2s^2(1-r^2)} \left(r_1^2 + r_2^2 + (1 - r^2) \right) \right)$$
$$\exp\left(-\frac{1}{2s^2(1-r^2)} \left(-2 r r_1 r_2 \cos(j_1 - j_2) - 2 r_1 (1 - r^2) \cos j_1 \right) \right) \quad (15)$$

Using the following transformations

$$X_1 = \frac{X^2 + Y^2}{2s^2(1-r^2)} = \frac{R_1^2}{2s^2(1-r^2)},\ X_2 = \frac{V^2 + Z^2}{2s^2(1-r^2)} = \frac{R_2^2}{2s^2(1-r^2)}, \quad (16)$$

the joint probability density function of X_1, X_2, Φ_1, $\Phi_2$is given as follows:

$$f_{X_1 X_2 \Phi_1 \Phi_2}(x_1, x_2, j_1, j_2) = \left(\frac{1}{2p} \right)^2 (1 - r^2)$$
$$\exp\left(-\left(x_1 + x_2 + \frac{1}{2s^2} - 2r\sqrt{x_1 x_2}\cos(j_1 - j_2) - 2\frac{\sqrt{x_1(1-r^2)}}{\sqrt{2s^2}}\cos j_1 \right) \right) \quad (17)$$

From this, the joint probability density function of X_1, X_2 can be described with the following formula after integration of Φ_1, Φ_2:

$$f_{X_1 X_2}(x_1, x_2) = (1 - r^2)\exp(-(x_1 + x_2 + g)) I_0\left(2|r|\sqrt{x_1 x_2}\right) I_0\left(2\sqrt{g(1-r^2)x_1}\right), \quad (18)$$

where $I_0()$ is the zero order modified Bessel function of the first kind, $\gamma = E_b/N_0 = 1/2\sigma^2$ is the signal-to-noise ratio.

The correlation peak is false detected, if the value of X_2 is bigger than X_1. The probability of this event is defined as follows:

$$\Pr(X_1 < X_2) = \int_0^\infty \int_{x_1}^\infty f_{X_1 X_2}(x_1, x_2) dx_2 dx_1 =$$
$$\int_0^\infty (1 - r^2)\exp(-x_1) I_0\left(2\sqrt{g(1-r^2)x_1}\right) \int_{x_1}^\infty \exp(-(x_2 + g)) I_0\left(2|r|\sqrt{x_1 x_2}\right) dx_2 dx_1. \quad (19)$$

This formula can be calculated using [17]:

$$\int_x^\infty \exp(-(x_2 + g)) I_0\left(2|r|\sqrt{x x_2}\right) dx_2 = \exp(-(x + g)) \sum_{k=0}^\infty |r|^k I_k(2|r|x), \quad (20)$$

then one can get the following result:

$$\Pr(X_1 < X_2) = \int_0^{\infty}\int_x^{\infty} f_{XX_2}(x, x_2)dx_2 dx =$$
$$\sum_{k=0}^{\infty}\left(1-r^2\right)|r|^k \exp(-g)\int_0^{\infty}\exp(-2x)I_0\left(2\sqrt{g\left(1-r^2\right)x}\right)I_k\left(2|r|x\right)dx, \tag{21}$$

where $I_k()$ is the kth order modified Bessel function of the first kind.

To simplify the previous equation, the Taylor series of the zero order Bessel function is used:

$$\int_0^{\infty}\exp(-2x)I_0\left(2\sqrt{g\left(1-r^2\right)x}\right)I_k\left(2|r|x\right)dx =$$
$$\sum_{l=0}^{\infty}\frac{\left(g\left(1-r^2\right)\right)^l}{l!\Gamma(l+1)}\int_0^{\infty}\exp(-2x)x^l I_k\left(2|r|x\right)dx. \tag{22}$$

Now, the following formula can be used from [18]:

$$\int_0^{\infty}\exp\left(-t\frac{z}{\sqrt{z^2-1}}\right)I_m(t)t^n\,dt = \frac{\Gamma(n+m+1)}{\left(z^2-1\right)^{-\frac{1}{2}(n+1)}}P_n^{-m}(z), \; -\operatorname{Re}(n+m)<1,$$
$$\text{if } t = 2|r|x, n = l, m = k, z = \frac{1}{\sqrt{1-r^2}}, \tag{23}$$

where $P_v^{-\mu}()$is the Legendre function of the third type with parameters v, μ. Using this, one can get:

$$\int_0^{\infty}\exp(-2x)x^l I_k\left(2|r|x\right)dx = \int_0^{\infty}\exp\left(-\frac{t}{|r|}\right)I_k(t)\frac{t^l}{\left(2|r|\right)^{l+1}}dt =$$
$$\frac{1}{\left(2|r|\right)^{l+1}}\frac{\Gamma(k+l+1)}{\left(\frac{r^2}{1-r^2}\right)^{-\frac{1}{2}(l+1)}}P_l^{-k}\left(\frac{1}{\sqrt{1-r^2}}\right) \tag{24}$$

Substituting in (19), the decision error probability is the following:

$$\Pr(X_1 < X_2) = \int_0^{\infty}\int_x^{\infty} f_{XX_2}(x, x_2)dx_2 dx =$$
$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{|r|^k}{2^{l+1}}\left(1-r^2\right)^{\frac{1}{2}(l+1)} g^l \exp(-g)\frac{(k+l)!}{(l!)^2}P_l^{-k}\left(\frac{1}{\sqrt{1-r^2}}\right) \tag{25}$$

3.3 Joint probability density function of the correlators' output signals and the probability of false decision using non-coherent sliding correlator

If the measurements of the correlation values belonging to *0* and τ delays are performed in non-overlapping time windows, then the notations (12) of the output signals given by (9) are going to be modified according to the independency of the two complex Gauss distributed probability variable $v_1(0)$ and $v_1(\tau)$as follows:

$$\begin{gathered}\Re(v_1(0)) = X = 2E_b + \sqrt{2E_b}n_{01},\ \mathrm{E}(X) = 2E_b,\ \mathrm{E}\left(\left(X - \mathrm{E}(X)^2\right)\right) = 2E_bN_0 = \sigma^2,\\ \Im(v_1(0)) = Y = \sqrt{2E_b}n_{02},\ \mathrm{E}(Y) = 0,\ \mathrm{E}\left(\left(Y - \mathrm{E}(Y)^2\right)\right) = 2E_bN_0 = \sigma^2,\\ \Re(v_1(\tau)) = V = 2E_b\rho + \sqrt{2E_b}n_{\tau 1},\ \mathrm{E}(V) = 2E_b\rho,\ \mathrm{E}\left(\left(V - \mathrm{E}(V)^2\right)\right) = 2E_bN_0 = \sigma^2,\\ \Im(v_1(\tau)) = Z = \sqrt{2E_b}n_{\tau 2},\ \mathrm{E}(Z) = 0,\ \mathrm{E}\left(\left(Z - \mathrm{E}(Z)^2\right)\right) = 2E_bN_0 = \sigma^2,\\ \mathrm{E}(XY) = 0,\ \mathrm{E}(VZ) = 0,\\ \mathrm{E}((X - \mathrm{E}(X))(V - \mathrm{E}(V))) = 0,\ \mathrm{E}((Y - \mathrm{E}(Y))(Z - \mathrm{E}(Z))) = 0.\end{gathered} \tag{26}$$

Using this, the four variable joint probability density function can be determined (if $T_c = 1$ *and* $2E_b = 1$):

$$f_{XYVZ}(x,y,v,z) = \left(\frac{1}{2\pi\sigma^2}\right)^2 \exp\left(-\frac{1}{2\sigma^2}\left((x-1)^2 + (v-\rho)^2 + y^2 + z^2\right)\right). \tag{27}$$

Using the transformation given by (14), one can get the following four-dimension joint probability density function:

$$\begin{aligned}&f_{R_1R_2\Phi_1\Phi_2}(r_1,r_2,\varphi_1,\varphi_2) = r_1,r_2\left(\frac{1}{2\pi\sigma^2}\right)^2 \exp\left(-\frac{1}{2\sigma^2}\left((r_1\cos\varphi_1 - 1)^2 + (r_2\cos\varphi_2 - \rho)^2\right)\right)\\ &\exp\left(-\frac{1}{2\sigma^2}\left((r_1\sin\varphi_1)^2 + (r_2\sin\varphi_2)^2\right)\right) =\\ &r_1,r_2\left(\frac{1}{2\pi\sigma^2}\right)^2 \exp\left(-\frac{1}{2\sigma^2}\left(r_1^2 + r_2^2 + (1+\rho^2) - 2r_1\cos\varphi_1 - 2\rho r_2\cos\varphi_2\right)\right),\end{aligned} \tag{28}$$

and using the following transformations:

$$X_1 = \frac{X^2 + Y^2}{2\sigma^2} = \frac{R_1^2}{2\sigma^2},\ X_2 = \frac{V^2 + Z^2}{2\sigma^2} = \frac{R_2^2}{2\sigma^2}, \tag{29}$$

one is able to calculate the following joint density function:

$$\begin{aligned}&f_{X_1X_2\Phi_1\Phi_2}(x_1,x_2,\varphi_1,\varphi_2) = \left(\frac{1}{2\pi}\right)^2\\ &\exp\left(-\left(x_1 + x_2 + \frac{1+\rho^2}{2\sigma^2} - 2\sqrt{\frac{x_1}{2\sigma^2}}\cos\varphi_1 - 2\rho\sqrt{\frac{x_2}{2\sigma^2}}\cos\varphi_2\right)\right).\end{aligned} \tag{30}$$

From this, the joint probability density function ofX_1 and X_2 variables is after integrating φ_1 and $\varphi_2$as follows:

$$f_{X_1X_2}(x_1,x_2) = \exp\left(-\left(x_1 + x_2 + (1+\rho^2)\gamma\right)\right) I_0\left(2\sqrt{\gamma x_1}\right) I_0\left(2|\rho|\sqrt{\gamma x_2}\right), \tag{31}$$

where $I_0()$ is the zero order modified Bessel function of the first kind and $\gamma = E_b/N_0 = 1/2\sigma^2$ is the signal-to-noise ratio.

The probability of the correlation peak's false detection is given by the following formula:

$$\begin{aligned} &\Pr(X_1 < X_2) = \int_0^{\infty}\int_{x_1}^{\infty} f_{X_1X_2}(x_1, x_2)dx_2dx_1 = \\ &\int_0^{\infty} \exp\left(-x_1 - \left(1 + r^2\right)g\right) I_0\left(2\sqrt{gx_1}\right)\int_{x_1}^{\infty}\exp\left(-(x_2)\right) I_0\left(2|r|\sqrt{gx_2}\right)dx_2dx_1. \end{aligned} \tag{32}$$

The second integral of the previous equation can be calculated using [17] as follows:

$$\int_x^{\infty}\exp\left(-(x_2)\right) I_0\left(2|r|\sqrt{gx_2}\right)dx_2 = \exp(-x)\sum_{k=0}^{\infty}\left(|r|\sqrt{\frac{g}{x}}\right)^k I_k\left(2|r|\sqrt{gx}\right). \tag{33}$$

Using this, the decision error probability is the following:

$$\begin{aligned} &\Pr(X_1 < X_2) = \int_0^{\infty}\int_{x}^{\infty} f_{XX_2}(x, x_2)dx_2dx = \\ &\sum_{k=0}^{\infty}\exp\left(-\left(1 + r^2\right)g\right)\int_0^{\infty}\exp(-2x)\left(|r|\sqrt{\frac{g}{x}}\right)^k I_0\left(2\sqrt{gx}\right) I_k\left(2|r|\sqrt{gx}\right)dx. \end{aligned} \tag{34}$$

4 Statistical analysis of the coherent system

This section introduces the system model using coherent receiving, the joint PDF of the correlators' output signals and the decision error probability for both seeking-and-tracking and sliding correlators.

4.1 Coherent synchronization system using training sequence

It is assumed in the following that the phase position of the signal is known at the time of the symbol synchronization, therefore the low-pass equivalents of the input signals have only real values and only the real part of the low-pass equivalent of the additive Gauss noise has effect on the correlators' output signals.

The coherent synchronization system using training sequence is depicted in Figure 5. The goal remains the same, i.e. the receiver wants to detect the correlation function maximum at $\tau = 0$ position, thereby determine the maximum position. This means coherent measuring of the correlation function's values belonging to 0 and τ delays and deciding on the greater of them. It is supposed that the transmitter sends a training sequence during the synchronization, e.g. sending $g_{c1}(t)$ to the receiver, as long as the synchronization is not finished. Note that the synchronization is possible without training sequence in this case too. The statistical properties of this system are analyzed in the two scenarios given in Figure 3a and Figure 3b.

The synchronposition is determined by choosing the greater signal from the result of the two correlation calculation. The null comparator is used for this purpose by deciding the sign of the difference between the correlators' output signal as shown in Figure 5.

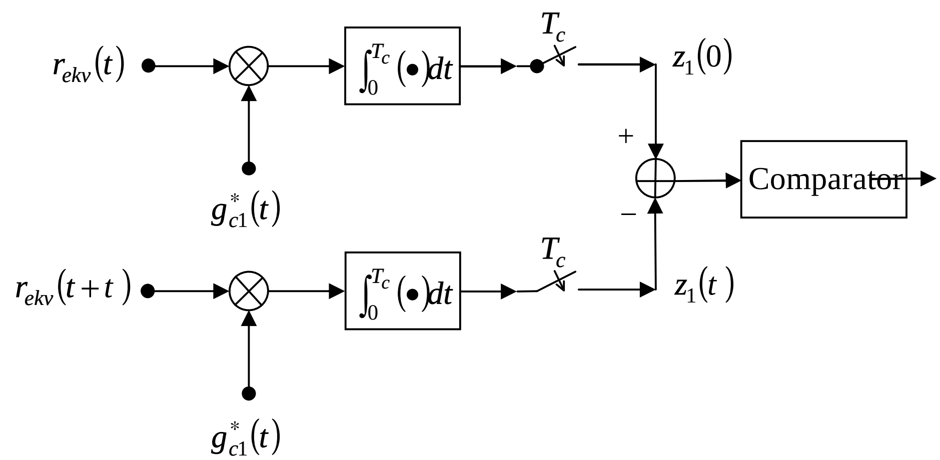


Figure 5 Coherent synchronization system using training sequence

The operation of the synchronizing system is given with the following equations:

$$z_1(\tau)=\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} r_{ekv}(t+\tau)g_{c1}^{*}(t)dt=\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} r_{ekv}(t+\tau)A\exp\left(-j2\pi\frac{\Delta f}{2T_c}t^2\right)dt, \qquad (35)$$

where

$$r_{ekv}(t+\tau)=Ag_1(t+\tau)+Ag_1(t-T_c+\tau)+n(t+\tau), \qquad (36)$$

and $z_1(\tau)$ is the output of the integrator at the end of the integration time, $n(t)$ is a Gauss distributed stochastic process with zero expected value and N_0 power density. The integration domains have $T_c/2$ offset for practical reasons.

Now, the task is to determine the probability of $z_1(0)<z_1(\tau)$ with given signal-to-noise ratio ($\gamma = E_b/N_0$), i.e. according to the appropriate synchronposition the τ value is chosen based on the given measurement. Using the simplification made in (5), the effect of the second part of (36) can be neglected because of the correlation properties of the chirp signal and we can get the following formula:

$$z_1(\tau)=A^2\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-\tau} g_{c1}(t+\tau)g_{c1}^{*}(t)dt+A\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+\tau)g_{c1}^{*}(t)dt=$$

$$A^2\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}-\tau}\exp\left(j2\pi\frac{\Delta f}{2T_c}(t+\tau)^2\right)\exp\left(-j2\pi\frac{\Delta f}{2T_c}t^2\right)dt+ \qquad (37)$$

$$A\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} n(t+\tau)\exp\left(-j2\pi\frac{\Delta f}{2T_c}t^2\right)dt.$$

Based on this, the two correlation values are as follows:

$$z_1(0) = 2E_b + \sqrt{2E_b}\, n_0, \qquad (38)$$
$$z_1(\tau) = 2E_b \rho + \sqrt{2E_b}\, n,$$

where ρ is given in equation (10).

4.2 Joint probability density function of the correlators' output signals and the probability of false decision using coherent seeking-and-tracking correlators

The goal is to calculate the joint probability density function of $z_1(0)$ and $z_1(\tau)$ defined by (38). To do that the following notations are introduced:

$$z_1(0) = X_1 = 2E_b + \sqrt{2E_b}\, n_0,\ \mathrm{E}(X_1) = 2E_b,\ \mathrm{E}\left(\left(X_1 - \mathrm{E}(X_1)\right)^2\right) = 2E_b N_0 = \sigma^2,$$
$$z_1(\tau) = X_2 = 2E_b \rho + \sqrt{2E_b}\, n_\tau,\ \mathrm{E}(X_2) = 2E_b \rho,\ \mathrm{E}\left(\left(X_2 - \mathrm{E}(X_2)\right)^2\right) = 2E_b N_0 = \sigma^2, \qquad (39)$$
$$\mathrm{E}\left(\left(X_1 - \mathrm{E}(X_1)\right)\left(X_2 - \mathrm{E}(X_2)\right)\right) = 2E_b N_0 \rho .$$

The bivariate joint PDF is given as follows (if $T_c = 1$ and $2E_b = 1$):

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho^2)}} \qquad (40)$$
$$\exp\left(-\frac{1}{2\sigma^2(1-\rho^2)}\left((x_1-1)^2 - 2\rho(x_1-1)(x_2-\rho) + (x_2-\rho)^2\right)\right).$$

With the following transformations:

$$U = X_1 - 1,\ V = \frac{X_2 - \rho - \rho(X_1 - 1)}{\sqrt{1-\rho^2}}, \qquad (41)$$

one is able to get:

$$f_{UV}(u, v) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left(u^2 + v^2\right)\right). \qquad (42)$$

The false decision probability is the following:

$$\Pr(X_1 < X_2) = \Pr\left(U < V\sqrt{\frac{1+\rho}{1-\rho}} - 1\right). \qquad (43)$$

Using this, one can get the following equivalent expression:

$$\Pr(X_1 < X_2) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{\gamma}{2}(1-\rho)}\right). \qquad (44)$$

4.3 Joint probability density function of the correlators' output signals and the probability of false decision using coherent sliding correlator

If the measurements of the correlation values belonging to *0* and *τ* delays are performed in non-overlapping time windows, then the notations (39) of the output signals given by (38) are going to be modified according to the independency of the two complex Gauss distributed probability variable $v_1(0)$ and $v_1(\tau)$as follows:

$$X_1 = 2E_b + \sqrt{2E_b}n_0,\ \mathrm{E}(X_1) = 2E_b,\ \mathrm{E}\left(\left(X_1 - \mathrm{E}(X_1)^2\right)\right) = 2E_b N_0 = s^2,$$
$$X_2 = 2E_b r + \sqrt{2E_b}n_t,\ \mathrm{E}(X_2) = 2E_b r,\ \mathrm{E}\left(\left(X_2 - \mathrm{E}(X_2)^2\right)\right) = 2E_b N_0 = s^2, \qquad (45)$$
$$\mathrm{E}\left((X_1 - \mathrm{E}(X_1))(X_2 - \mathrm{E}(X_2))\right) = 0.$$

Using this, the bivariate joint PDF is given as follows:

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2ps^2}\exp\left(-\frac{1}{2s^2}\left((x_1-1)^2 + (x_2 - r)^2\right)\right). \qquad (46)$$

Introducing the following transformations:

$$U = X_1 - 1, V = X_2 - r, \qquad (47)$$

one can get the following density function:

$$f_{UV}(u,v) = \frac{1}{2ps^2}\exp\left(-\frac{1}{2s^2}\left(u^2 + v^2\right)\right). \qquad (48)$$

The false decision probability based on that is:

$$\Pr(X_1 < X_2) = \Pr\left(U < V - (1 - r)\right). \qquad (49)$$

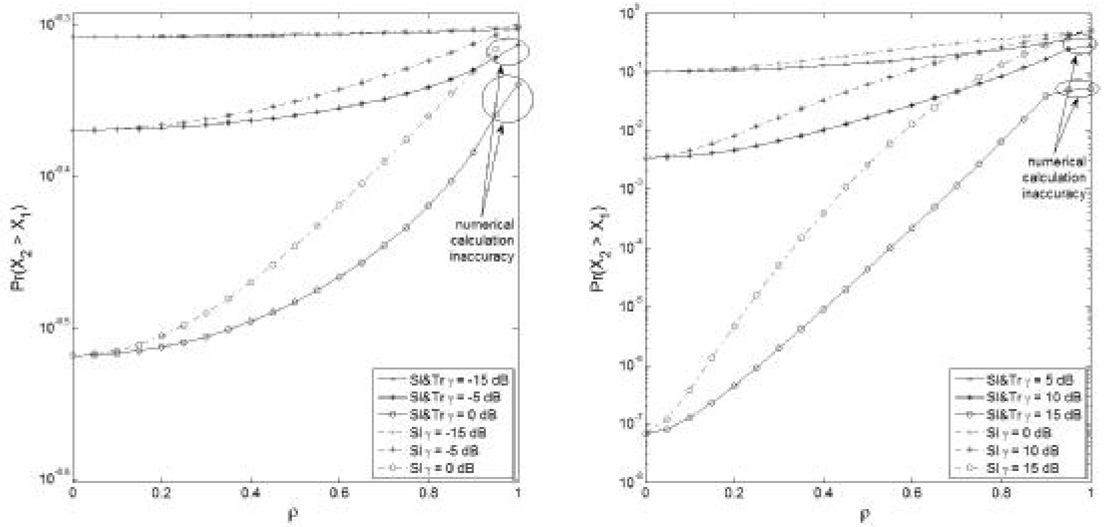
From this, the final expression is determined as:

$$\Pr(X_1 < X_2) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{g}{2}(1-r)^2}\right). \qquad (50)$$

5 Numerical results

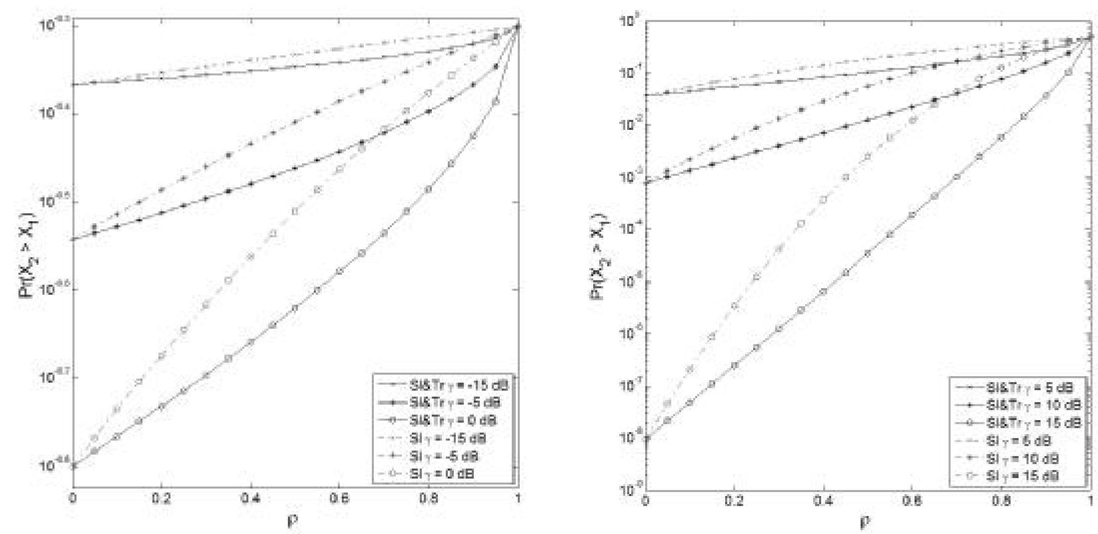
This section presents the numerical results. Each figure depicts the comparison of the sliding and the seeking-and-tracking correlators depending on different parameters. Figure 6 shows the decision error probability in function of the correlation parameter using non-coherent receiving assuming low (Figure 6a) and high (Figure 6b) signal-to-noise ratios. It is clearly visible that the decision error probability is lower with seeking-and-tracking correlators than with simple sliding correlator. It should be noted, that there are some inaccuracy when ρ is close to *1* due to complicated numerical calculations (see (25) and (34)).

Similar results are represented by Figure 7. This figure indicates the decision error probability in function of the correlation parameter using coherent receiving when the SNR equals to *-15 – 0 dB*(Figure 7a) and *5–15 dB* (Figure 7b). In coherent receiving case the proposed seeking-and-tracking correlators achieve much lower error probabilities than the simple sliding correlator.



a) When E_b/N_0 = *-15 – 0 dB* **b)** When E_b/N_0 = *5 – 15 dB*

Figure 6 The decision error probability in function of the correlation parameter using non-coherentreceiving

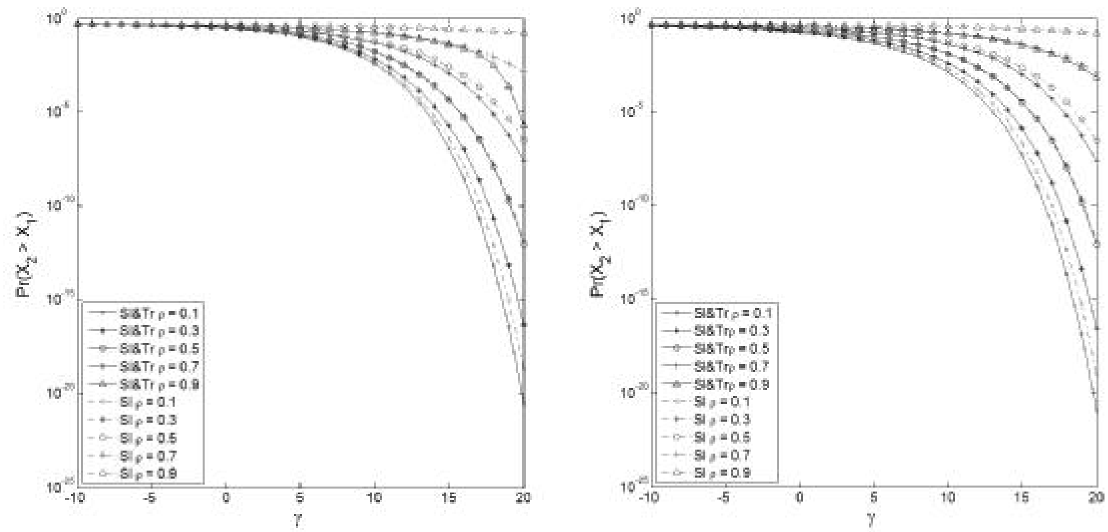


a) When E_b/N_0 = *-15 – 0 dB* **b)** When E_b/N_0 = *5 – 15 dB*

Figure 7 The decision error probability in function of the correlation parameter using coherent receiving

Figure 8 depicts the decision error probability depending on signal-to-noise ratio using non-coherent (Figure 8a) and coherent (Figure 8b) receiving in case of $\rho = 0.1 - 0.9$. It can be determined that the seeking-and-tracking correlators outperform the sliding correlator in both coherent and non-coherent cases.

The results (Figure 6 – Figure 8) basically tell, that if there are some dependency between the two measurements, i.e. the noises are not statistically independent, then the seeking-and-tracking correlators are able to use this dependency more effectively to reduce the probability of the wrong decision than the sliding correlator.



a) Using non-coherent receiving **b)** Using coherent receiving

Figure 8 The decision error probability in function of signal-to-noise ratio in case of $\rho = 0.1 – 0.9$

6 Conclusion

In this paper a new, so-called seeking-and-tracking correlators were proposed and the statistical analysis of the error probability of correlation peak detection using AWGN channel was provided in such systems, in which the autocorrelation function is unimodal, i.e. spectrum spread based systems like DS and FFH. The seeking-and-tracking correlators have the advantage of using the statistically not independent noises to reduce the error probability of finding the correlation peak. It was shown by numerical results that the seeking-and-tracking correlators outperform the conventional sliding correlator in every investigated case. The presented formulas can be easily adapted for other spectrum spread based systems to achieve more accurate synchronization, thus emerging UWB and WLAN positioning systems can profit from the results of this paper.

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