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Sheaf Completeness of Quantum Logic

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Abstract: Quantum logic is a counterpart of orthomodular lattice and interpreted in a sheaf of complete Boolean algebra. This means that quantum logic is interpreted in a family of classical worlds. The aim of this paper is to prove a sheaf completeness of quantum logic.

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1 Introduction

Quantum theory is described in the language represented on the orthomodular lattice which consists of projections on a Hilbert space.

Ortholattice is a lattice provided with an operator $^\perp$ such that

(**C1**) $a^{\perp\perp} = a$;

(**C2**) $a \vee a^\perp = 1, \quad a \wedge a^\perp = 0$;

(**C3**) $a \le b \Longrightarrow b^\perp \le a^\perp$.

Orthomodular lattice is an ortholattice satisfying orthomodularity:

(**P**) $a \le b \implies b = a \vee (b \wedge a^\perp)$.

Quantum logic is a counterpart of orthomodular lattice, which is interpreted in a sheaf of complete Boolean algebra. That is, each formula of quantum logic is interpreted as a cross-section of a sheaf whose stalks are complete Boolean algebras. This means that quantum logic is interpreted in a family of classical worlds. Accordingly the set theory based on the quantum logic is represented in a sheaf of classical universe. (cf.Titani[12])

Aim of this paper is to prove the sheaf completeness of quantum logic.

2 Quantum logic $\mathrm{QL}_\Box$

In this paper we deal with the system $\mathrm{QL}_\Box$ of quantum logic which is introduced in Titani-Kodera-Aoyama [13]. $\mathrm{QL}_\Box$ is a predicate orthologic with

$$\textbf{primitive logical symbols} : \wedge,\ \vee,\ ^\perp,\ \exists,\ \forall,\ \Box,$$

and with additional inference rule of orthomodularity, where we referred to Birkhoff [1], [2], Gentzen [3], Piron [5], Takeuti [10], [11].

$\Box$ is interpreted on a orthomodular lattice as

$$\Box a = \begin{cases} 1 & a = 1 \\ 0 & \text{otherwise} \end{cases}$$

Finite sequences of formulas are denoted by Γ, Δ, $\cdots$. If Γ is a sequence "$\varphi_1, \cdots, \varphi_n$" of formulas, then the sequence "$\Box\varphi_1, \cdots, \Box\varphi_n$" is denoted by $\Box\Gamma$.

Logical axioms are sequents of the form $\varphi \Rightarrow \varphi$.

Structural rules :

Thinning : $\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

Contraction : $\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ $\qquad$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$

Interchange : $\dfrac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}$ $\qquad$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$

Cut : $\dfrac{\Gamma \Rightarrow \Box\Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Box\Delta, \Lambda}$ $\qquad$ $\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Box\Pi \Rightarrow \Lambda}{\Gamma, \Box\Pi \Rightarrow \Delta, \Lambda}$

$$\frac{\Gamma \Rightarrow \Delta, \Box\varphi \quad \Box\varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical rules:

$\wedge$: $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$ $\qquad$ $\dfrac{\Gamma \Rightarrow \Box\Delta, \varphi \quad \Gamma \Rightarrow \Box\Delta, \psi}{\Gamma \Rightarrow \Box\Delta, \varphi \wedge \psi}$

$\dfrac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$ $\qquad$ $\dfrac{\Gamma \Rightarrow \Delta, \Box\varphi \quad \Gamma \Rightarrow \Delta, \Box\psi}{\Gamma \Rightarrow \Delta, \Box\varphi \wedge \Box\psi}$

$\vee$:
$$\frac{\varphi, \Box\Gamma \Rightarrow \Delta \quad \psi, \Box\Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Box\Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Box\varphi, \Gamma \Rightarrow \Delta \quad \Box\psi, \Gamma \Rightarrow \Delta}{\Box\varphi \vee \Box\psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$\forall$:
$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Box\Delta, \varphi(a)}{\Gamma \Rightarrow \Box\Delta, \forall x\varphi(x)} \quad \frac{\Gamma \Rightarrow \Delta, \Box\varphi(a)}{\Gamma \Rightarrow \Delta, \Box\forall x\Box\varphi(x)}$$

where t is any term; where a is a free variable which does not occur in the lower sequent.

$\exists$:
$$\frac{\varphi(a), \Box\Gamma \Rightarrow \Delta}{\exists x\varphi(x), \Box\Gamma \Rightarrow \Delta} \quad \frac{\Box\varphi(a), \Gamma \Rightarrow \Delta}{\Diamond\exists x\Box\varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\varphi(x)}$$

where a is a free variable which does not occur in the lower sequent; where t is any term

$(C1)$:
$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi^{\perp\perp}, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi^{\perp\perp}}$$

$(C2)$:
$$\frac{\Gamma \Rightarrow \Box\Delta, \varphi}{\varphi^{\perp}, \Gamma \Rightarrow \Box\Delta} \quad \frac{\Gamma \Rightarrow \Delta, \Box\varphi}{(\Box\varphi)^{\perp}, \Gamma \Rightarrow \Delta} \quad \frac{\varphi, \Box\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Delta, \varphi^{\perp}} \quad \frac{\Box\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, (\Box\varphi)^{\perp}}$$

$(C3)$:
$$\frac{\varphi, \Box\Gamma \Rightarrow \Box\Delta, \psi}{\psi^{\perp}, \Box\Gamma \Rightarrow \Box\Delta, \varphi^{\perp}}$$

(P) :
$$\frac{\varphi, \Box\Gamma \Rightarrow \Box\Delta, \psi}{\psi, \Box\Gamma \Rightarrow \Box\Delta, \varphi \vee (\psi \wedge \varphi^{\perp})} \qquad \text{[Orthomodularity]}$$

$(\Box)$:
$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Box\Gamma \Rightarrow \Box\Delta, \varphi}{\Box\Gamma \Rightarrow \Box\Delta, \Box\varphi} \qquad \text{[Globalization].}$$

Defined formulas are :

$$\top \overset{\text{def}}{\Longleftrightarrow} (\varphi \vee \varphi^{\perp})$$
$$\perp \overset{\text{def}}{\Longleftrightarrow} (\top)^{\perp}$$
$$\varphi \rightarrow_{\mathbf{T}} \psi \overset{\text{def}}{\Longleftrightarrow} \varphi^{\perp} \vee (\varphi \wedge \psi) \qquad \text{(cf.Takeuti[9],[10])}$$
$$\varphi \supset \psi \overset{\text{def}}{\Longleftrightarrow} \Box(\varphi \rightarrow_{\mathbf{T}} \psi)$$
$$\neg\varphi \overset{\text{def}}{\Longleftrightarrow} \varphi \supset \perp$$
$$\varphi \leftrightarrow \psi \overset{\text{def}}{\Longleftrightarrow} (\varphi \supset \psi) \wedge (\psi \supset \varphi)$$
$$\Diamond\varphi \overset{\text{def}}{\Longleftrightarrow} (\Box\varphi^{\perp})^{\perp}$$
$$\varphi \,\text{\textdownarrow}\, \psi \overset{\text{def}}{\Longleftrightarrow} \Box\big((\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \vee (\varphi^{\perp} \wedge \psi) \vee (\varphi^{\perp} \wedge \psi^{\perp})\big)$$

3 Sheaf models

Definition 3.1. Let X be a topological space and B_x be a complete Boolean algebra, called **stalk**, for each $x \in X$. Then the direct product of stalks

$$\prod_{x\in U} B_x = \{f : U \to \bigcup_{x\in U} B_x \mid f(x) \in B_x\}$$

is a complete Boolean algebra for each open set $U \in \mathcal{O}(X)$, where

$$f \leq g \Longleftrightarrow \forall x \in U(f(x) \leq g(x)), \qquad f^{\perp}(x) = f(x)^{\perp}$$

(cf.Sikorski[7]). Then a pair $\langle F, r\rangle$ satisfying the following conditions is called a **sheaf of complete Boolean algebra** over X.

(1) F is a mapping which associates a complete Boolean sub-algebra $F(U)$ of $\prod_{x\in U} B_x$ to each $U \in \mathcal{O}(X)$.

(2) If $U, V \in \mathcal{O}(X)$ and $U \subset V$, then there exists a homomorphism $r_{U,V} : F(V) \to F(U)$ preserving $\bigvee$ and $^{\perp}$, i.e.

$$r_{U,V}(\bigvee_i f_i) = \bigvee_i (r_{U,V}(f_i)), \quad r_{U,V}(f^{\perp}) = (r_{U,V}(f))^{\perp}.$$

(3) $F(\emptyset) = 0, \quad r_{U,U} = 1$.

(4) If $U, V, W \in \mathcal{O}(X)$ and $U \subset V \subset W$, then

$$r_{U,W} = r_{U,V} \circ r_{V,W}.$$

(5) If $\{U_i\}_{i\in I} \in \mathcal{O}(X)$ and $U = \bigcup_i U_i$ and further if

$$\forall i\big(f_i \in F(U_i)\big) \ \wedge \ \forall i, j \in I\big(r_{U_i\cap U_j, U_i}(f_i) = r_{U_i\cap U_j, U_j}(f_j)\big),$$

then there exists a unique $f \in F(U)$ such that $\forall i \in I\big(r_{U_i,U}(f) = f_i\big)$.

$\Gamma(U, f) = \{\langle x, f(x)\rangle \mid x \in U\}$ is a cross section of sheaf $\langle F, r\rangle$.

A **sheaf model** of $\mathrm{QL}_{\Box}$ is a triple $\mathcal{M} = \langle \mathcal{F}, D, I\rangle$, where :
$\mathcal{F}$ is a set of cross sections of a sheaf.
D is a nonempty set, which is a domain of free variables.
I is an interpretation of predicate constants :

$$I(p) : D^n \to \mathcal{F} \quad \text{for each } n\text{-ary predicate constant } p.$$

A D**-assignment** v is a map on the set FV of free variables to D:

$$v : FV \to D$$

A pair $\langle \mathcal{M}, v\rangle$ of sheaf model $\mathcal{M}$ and a D-assignment v determines a truth value of each formula in $\mathcal{F}$.

The truth value of a formula φ in $\langle \mathcal{M}, v\rangle$ is denoted by $\varphi[\mathcal{M}, v] \in \mathcal{F}$. If the truth value of formula φ is $\Gamma(U, f)$, then U is denoted by $E[\![\varphi]\!]$ and f is denoted by $f_{[\![\varphi]\!]}$.

$$\varphi[\mathcal{M}, v] = \Gamma(E[\![\varphi]\!], f_{[\![\varphi]\!]}) \in \mathcal{F}$$

Definition 3.2. A formula φ is said to be **valid** in $\langle \mathcal{M}, v\rangle$ if $E[\![\varphi]\!] = X$ and $f_{[\![\varphi]\!]} = 1$, and denoted by

$$\langle \mathcal{M}, v\rangle \models \varphi.$$

If $\langle \mathcal{M}, v\rangle \models \varphi$ for all $\langle \mathcal{M}, v\rangle$, then φ is said to be valid and denoted by

$$\models \varphi.$$

A sequent $\varphi_1, \cdots, \varphi_m \Rightarrow \psi_1, \cdots, \psi_n$ is **valid**, if

$$\models \varphi_1 \wedge \cdots \wedge \varphi_m \supset \psi_1 \vee \cdots \vee \psi_n,$$

Theorem 3.1 (Soundness)**.** *If φ is a provable formula $\mathrm{QL}_\Box$, then valid, i.e.*

$$\mathrm{QL}_\Box \vdash \varphi \implies \models \varphi.$$

4 Sheaf completeness of $\mathrm{QL}_\Box$

" $\mathrm{QL}_\Box$ is **sheaf complete**" means that " if a formula φ of $\mathrm{QL}_\Box$ is valid in all sheaf models, then φ is provable in $\mathrm{QL}_\Box$ " :

$$\models \varphi \implies \mathrm{QL}_\Box \vdash \varphi.$$

In order to prove the sheaf completeness of $\mathrm{QL}_\Box$, we suppose that φ_0 is a non provable formula:

$$\mathrm{QL}_\Box \not\vdash \varphi_0$$

and construct a sheaf model $\mathcal{M}$ and an assignment v of free variables such that φ_0 is not valid in $\langle \mathcal{M}, v\rangle$:

$$\langle \mathcal{M}, v\rangle \not\models \varphi_0 .$$

4.1 Construction of orthomodular lattice Q

Let p and q be distinct 0-ary predicate symbols which do not occur in φ_0, and $\mathrm{QL}_\Box[FV]$ be the system obtained from $\mathrm{QL}_\Box$ by adding the set FV of free variables as constant symbols. Since $\mathrm{QL}_\Box$ is a predicate orthologic, we can define the same equivalence relation $\equiv$ on the set of sentences of $\mathrm{QL}_\Box[FV]$ as in Titani-Kodera-Aoyama[13], in the same way as Takano[8].

Let $|\varphi|$ be the equivalence class of φ and Q be the quotient lattice.

$$Q = \{|\varphi| \mid \varphi \text{ is a sentence of } \mathrm{QL}_\Box[FV]\}, \quad \text{where} \quad |\varphi| = \{\psi \in Q \mid \psi \equiv \varphi\}.$$

Then we have

Proposition 4.1.

(1) $|\varphi \wedge \psi| = |\varphi| \wedge |\psi|, \quad |\varphi \vee \psi| = |\varphi| \vee |\psi|, \quad |\varphi^\perp| = |\varphi|^\perp$

(2) $|\varphi \supset \psi| = \begin{cases} 1 & \textit{if } |\varphi| \le |\psi|, \\ 0 & \textit{otherwise}; \end{cases}$

(3) $|\forall x \varphi(x)| = \bigwedge\{|\varphi(a)| \mid a \in FV\};$

(4) $|\exists x\varphi(x)| = \bigvee\{|\varphi(a)| \mid a \in FV\}$, *where* $\bigwedge$ *and* $\bigvee$ *denote the infimum and the supremum, respectively, in* Q.

(5) $|\varphi|^{\perp\perp} = |\varphi|$;

(6) $|\varphi| \wedge |\varphi|^{\perp} = 0, \quad |\varphi| \vee |\varphi|^{\perp} = 1$;

(7) $|\varphi| \leq |\psi| \iff |\psi|^{\perp} \leq |\varphi|^{\perp}$.

(8) $|\varphi_0| \leq |p| \lneqq |q|$.

Proof. cf. Titani-Kodera-Aoyama [13]. □

Furthermore, by the inference rule (P) of $\mathrm{QL}_{\Box}$,

$$\mathrm{QL}_{\Box} \vdash \varphi \supset \psi \Rightarrow \varphi \,\rfloor_{\circ}\, \psi.$$

Hence, $|\varphi| \leq |\psi| \Longrightarrow |\varphi| \,\rfloor_{\circ}\, |\psi|$. It follows that

Theorem 4.2. $Q = \langle Q, \leq, \bigwedge, \bigvee, {}^{\perp} \rangle$ *is a countable orthomodular lattice.*

4.2 Extension $\mathcal{L}^{\perp\perp}(Q)$

Definition 4.1. If $\alpha \subset Q$, then $\alpha^{\perp}$ is the set of all elements of Q which is orthogonal to α.

$$\alpha^{\perp} \stackrel{\text{def}}{=} \{\xi \in Q \mid \forall a \in \alpha (\xi \leq a^{\perp})\}.$$

Lemma 4.3. *Let* $\alpha, \beta \subset Q$.

(1) $0 \in \alpha^{\perp}$;

(2) *If* $\alpha \subset \beta$, *then* $\beta^{\perp} \subset \alpha^{\perp}$;

(3) $\alpha \subset \alpha^{\perp\perp}$ *and* $\alpha^{\perp} = \alpha^{\perp\perp\perp}$;

(4) $\alpha \cap \alpha^{\perp} = \{0\}$;

(5) $(\alpha \cup \alpha^{\perp})^{\perp\perp} = Q$.

Definition 4.2. The set of all subsets of Q such that $\alpha = \alpha^{\perp\perp}$ is denoted by $\mathcal{L}^{\perp\perp}(Q)$.

$$\mathcal{L}^{\perp\perp}(Q) \stackrel{\text{def}}{=} \{\alpha \subset Q \mid \alpha = \alpha^{\perp\perp}\}.$$

Theorem 4.4 (cf.Titani-Kodera-Aoyama[13]; McNeille[4]). $\mathcal{L}^{\perp\perp}(Q)$ *is a complete ortholattice, where*

$$\alpha \leq \beta \iff \alpha \subset \beta, \qquad \bigvee_i \alpha_i = (\bigcup_i \alpha_i)^{\perp\perp}, \qquad \bigwedge_i \alpha_i = \bigcap_i \alpha_i.$$

Lemma 4.5. *If* $a \in Q$, *then*

(1) $\{a\}^{\perp\perp} = \{\xi \in Q \mid \xi \leq a\}$.

(2) $\{a\}^{\perp} = \{a^{\perp}\}^{\perp\perp}$.

Proof. Using Lemma 4.3,

(1) $\xi \in \{a\}^{\perp\perp} \iff \forall\eta \in \{a\}^{\perp}(\xi \leq \eta^{\perp})$ and $a^{\perp} \in \{a\}^{\perp}$,
$\therefore \quad \xi \in \{a\}^{\perp\perp} \Longrightarrow \xi \leq a^{\perp\perp} = a.$

$$\begin{aligned} \xi \leq a \Longrightarrow a^{\perp} \leq \xi^{\perp} \quad &\text{and} \quad \eta \in \{a\}^{\perp} \Longrightarrow \eta \leq a^{\perp} \\ \therefore \quad \xi \leq a \quad &\Longrightarrow \quad \forall\eta \in \{a\}^{\perp}(\xi \leq \eta^{\perp}) \\ &\Longrightarrow \quad \xi \in \{a\}^{\perp\perp} \end{aligned}$$

(2) $\xi \in \{a\}^{\perp} \iff \xi \leq a^{\perp}. \quad \therefore \quad \{a\}^{\perp} = \{a^{\perp}\}^{\perp\perp}$

□

Corollary 4.6. *Q is embedded into $\mathcal{L}^{\perp\perp}(Q)$ by $a \mapsto \{a\}^{\perp\perp}$.*

Lemma 4.7. *For $\alpha, \beta \subset Q$,*

$$\beta^{\perp\perp} \subset \alpha^{\perp\perp} \iff \forall c \in Q \left(\alpha \subset \{c\}^{\perp\perp} \Rightarrow \beta \subset \{c\}^{\perp\perp}\right).$$

Proof.

$$\begin{aligned} \left(\beta^{\perp\perp} \subset \alpha^{\perp\perp}\right) \wedge \left(\alpha \subset \{c\}^{\perp\perp}\right) \quad &\Longrightarrow \quad \beta \subset \beta^{\perp\perp} \subset \alpha^{\perp\perp} \subset \{c\}^{\perp\perp} \\ \therefore \quad \beta^{\perp\perp} \subset \alpha^{\perp\perp} \quad &\Longrightarrow \quad \forall c \in Q \left(\alpha \subset \{c\}^{\perp\perp} \Longrightarrow \beta \subset \{c\}^{\perp\perp}\right) \end{aligned}$$

$$\begin{aligned} \forall c \in Q \left(\alpha \subset \{c\}^{\perp\perp} \Longrightarrow \beta \subset \{c\}^{\perp\perp}\right) \wedge \left(x \in \alpha^{\perp}\right) \quad &\Longrightarrow \quad \forall y \in \alpha(x \leq y^{\perp}) \\ &\Longrightarrow \quad \forall y \in \alpha(y \leq x^{\perp}) \\ &\Longrightarrow \quad \forall z \in \beta(z \leq x^{\perp}) \\ &\Longrightarrow \quad \forall z \in \beta(x \leq z^{\perp}) \\ &\Longrightarrow \quad x \in \beta^{\perp} \\ \therefore \quad \forall c \in Q \left(\alpha \subset \{c\}^{\perp\perp} \Longrightarrow \beta \subset \{c\}^{\perp\perp}\right) \quad &\Longrightarrow \quad \alpha^{\perp} \subset \beta^{\perp} \\ \therefore \quad \forall c \in Q \left(\alpha \subset \{c\}^{\perp\perp} \Longrightarrow \beta \subset \{c\}^{\perp\perp}\right) \quad &\Longrightarrow \quad \beta^{\perp\perp} \subset \alpha^{\perp\perp} \end{aligned}$$

□

Lemma 4.8. *If $\{a_i\}_i \subset Q$ and $(\bigvee_i a_i) \in Q$, then $\{\bigvee_i a_i\}^{\perp\perp} = (\bigcup_i \{a_i\}^{\perp\perp})^{\perp\perp}$.*

Proof. $\{a_i\}^{\perp\perp} \subset \{\bigvee_i a_i\}^{\perp\perp}$ for all i. Hence, $(\bigcup_i \{a_i\}^{\perp\perp})^{\perp\perp} \subset \{\bigvee_i a_i\}^{\perp\perp}$.
If $\bigcup_i \{a_i\}^{\perp\perp} \subset \{c\}^{\perp\perp}$, then $\bigvee_i a_i \leq c$. Hence, $\{\bigvee_i a_i\}^{\perp\perp} \subset \{c\}^{\perp\perp}$. It follows by Lemma 4.7 that

$$(\textstyle\bigcup_i \{a_i\}^{\perp\perp})^{\perp\perp} = \{\bigvee_i a_i\}^{\perp\perp}.$$

□

4.3 Sheaf representation of ortholattice $\mathcal{L}^{\perp\perp}(Q)$

It will be shown in this section that $\mathcal{L}^{\perp\perp}(Q)$ is embedded in a set of cross-sections of a sheaf of complete Boolean algebra.

4.3.1 Base space $\langle X, \mathcal{O}(X)\rangle$

Definition 4.3. A subset p of Q is **(mutually) compatible** if

$$\forall \xi, \eta \in p\,(\,\xi \underset{\circ}{\downarrow} \eta\,).$$

Let X be the set of all maximal compatible subsets of Q. Then $Q = \bigcup_{x\in X} x$. We define

$$\mathcal{O}(X) \overset{\text{def}}{=} \{\textstyle\bigcup_{i\in I} U_{p_i} \mid \{p_i\}_i \subset \mathcal{P}(Q)\} \quad \text{where}$$

$$U_p \overset{\text{def}}{=} \{x \in X \mid p \subset x\} \qquad \text{for } p \subset Q.$$

Since $U_p \cap U_q = U_{p\cup q}$ for $p, q \subset Q$, $\{U_p \mid p \subset Q\}$ forms a base of topology, i.e. $X = \langle X, \mathcal{O}(X)\rangle$ is a topological space.

Lemma 4.9.

$$U_{\{0\}} = U_{\{1\}} = X.$$

4.3.2 Stalks

Each element of X is a Boolean sub-algebra of Q because of mutual compatibility (cf.Piron[5]), and extended to a complete Boolean algebra by the following theorem.

Theorem 4.10 (Minimal extension)**.** *For an arbitrary Boolean algebra B , there exists a complete Boolean algebra B^*, called* **minimal extension** *of B, with the canonical isomorphism h^* preserving all infinite joins and meets, i.e.*

$$\textit{if} \quad a = \textstyle\bigvee^{B}_{t\in T} a_t, \ \textit{then} \quad h^*(a) = \bigvee^{B^*}_{t\in T} h^*(a_t), \tag{4.1}$$

$$\textit{if} \quad a = \textstyle\bigwedge^{B}_{t\in T} a_t, \ \textit{then} \quad h^*(a) = \bigwedge^{B^*}_{t\in T} h^*(a_t). \tag{4.2}$$

(cf.Rasiowa-Sikorski[6].)

Definition 4.4. The minimal extension of $x \in X$ is denoted by x^*.

$$x^* : \text{ the minimal extension of } x \text{ for each } x \in X. \quad (\text{ Theorem 4.10})$$

x^* for each $x \in X$ is a complete Boolean algebra. The supremum and infimum of $p \subset x$ in x^* is denoted by $\bigvee^{x^*} p$ and $\bigwedge^{x^*} p$, respectively.

Lemma 4.11. *If $\alpha \in \mathcal{L}^{\perp\perp}(Q)$, then there exists a compatible $p \subset Q$ such that $\alpha = p^{\perp\perp}$, hence there exists $x \in X$ such that $p \subset x$ and $\alpha = p^{\perp\perp}$.*

$$\alpha \in \mathcal{L}^{\perp\perp}(Q) \implies \exists x \in X \exists p \subset x\big(\alpha = p^{\perp\perp}\big).$$

Proof. Enumerate α:

$$\alpha = \{\xi_1, \xi_2, \cdots\}.$$

$p = \{\xi_1,\ (\xi_1 \vee \xi_2), \cdots, (\xi_1 \vee \xi_2 \vee \xi_3), \cdots\}$ is mutually compatible, because of orthomodularity, and $p^{\perp\perp} = \alpha$. □

LEMMA 4.12. *If $p \subset x$, then $(p^{\perp\perp} \cap x)^{\perp\perp} = p^{\perp\perp}$.*

Proof. $p^{\perp\perp} \cap x \subset p^{\perp\perp} \quad \therefore\ (p^{\perp\perp} \cap x)^{\perp\perp} \subset p^{\perp\perp}$. Conversely, $p \subset x$ implies $p \subset p^{\perp\perp} \cap x$, hence $p^{\perp\perp} \subset (p^{\perp\perp} \cap x)^{\perp\perp}$. □

COROLLARY 4.13. *If $\alpha \in \mathcal{L}^{\perp\perp}(Q)$, then there exists $x \in X$ such that*

$$\alpha = (\alpha \cap x)^{\perp\perp}.$$

Proof. By Lemma 4.11, there exists $x \in X$ and $p \subset x$ such that $\alpha = p^{\perp\perp}$. Hence, by Lemma 4.12,

$$\alpha = p^{\perp\perp} = (p^{\perp\perp} \cap x)^{\perp\perp} = (\alpha \cap x)^{\perp\perp}.$$

□

LEMMA 4.14. *If $x \in X$ and $p \subset x$, then*

$$\bigvee^{x^*}(p^{\perp\perp} \cap x) = \bigvee^{x^*} p \quad \textit{hence} \quad p^{\perp\perp} \cap x = \{\xi \in x \mid \xi \leq \bigvee^{x^*} p\}.$$

Proof. $p \subset p^{\perp\perp} \cap x \quad \therefore\ \bigvee^{x^*} p \leq \bigvee^{x^*}(p^{\perp\perp} \cap x)$.

Conversely, if $\bigvee^{x^*} p \leq \xi$ for $\xi \in x$, then $p \subset \{\xi\}^{\perp\perp}$. Hence, $p^{\perp\perp} \subset \{\xi\}^{\perp\perp}$.

$$\therefore \quad \bigvee^{x^*}(p^{\perp\perp} \cap x) \leq \bigvee^{x^*}(\{\xi\}^{\perp\perp} \cap x) = \xi.$$

$$\therefore \quad \bigvee^{x^*}(p^{\perp\perp} \cap x) = \bigvee^{x^*} p.$$

□

COROLLARY 4.15. *If $x \in X$ and $p, q \subset x$, then*

$$p^{\perp\perp} \leq q^{\perp\perp} \iff \bigvee^{x^*}(p^{\perp\perp} \cap x) \leq \bigvee^{x^*}(q^{\perp\perp} \cap x).$$

Proof. Suppose $\bigvee^{x^*}(p^{\perp\perp} \cap x) \leq \bigvee^{x^*}(q^{\perp\perp} \cap x)$.
By Lemma 4.14, $(p^{\perp\perp} \cap x) \subset (q^{\perp\perp} \cap x)$. Using Lemma 4.12,

$$p^{\perp\perp} = (p^{\perp\perp} \cap x)^{\perp\perp} \leq (q^{\perp\perp} \cap x)^{\perp\perp} = q^{\perp\perp}.$$

$$\therefore\ \bigvee^{x^*}(p^{\perp\perp} \cap x) \leq \bigvee^{x^*}(q^{\perp\perp} \cap x) \implies p^{\perp\perp} \subset q^{\perp\perp}.$$

The converse is obvious. □

4.4 A sheaf model of $\mathrm{QL}_{\Box}$

Definition 4.5. If $U \in \mathcal{O}(X)$, the direct product of complete Boolean algebras $\{x^* \mid x \in U\}$

$$\prod_{x\in U} x^* = \{f : U \to \bigcup_{x\in U} x^* \mid f(x) \in x^*\}$$

is a complete Boolean algebra with respect to pointwise $\leq$ and $^{\perp}$:

$$f \leq g \overset{\text{def}}{\iff} \forall x \in U(f(x) \leq g(x)) \quad \text{for } f, g \in \prod_{x\in U} x^*.$$
$$f^{\perp}(x) \overset{\text{def}}{=} (f(x))^{\perp} \quad \text{for } x \in U \text{ and } f \in \prod_{x\in U} x^*.$$

If $f \in \prod_{x\in U} x^*$ and $V \subset U$, then the restriction of f on V is denoted by $f \restriction V$.

$$(f \restriction V)(x) \overset{\text{def}}{=} f(x) \quad \text{for } x \in V.$$

Definition 4.6. If $\alpha \in \mathcal{L}^{\perp\perp}(Q)$, there exists a compatible p such that $\alpha = p^{\perp\perp}$ by Lemma 4.11. We define $E\alpha \in \mathcal{O}(X)$ by

$$E\alpha \overset{\text{def}}{=} \bigcup\{U_p \in \mathcal{O}(X) \mid (p \text{ is compatible}) \wedge (\alpha = p^{\perp\perp})\},$$

where $U_p = \{x \in X \mid p \subset x\}$.

Corollary 4.16. $E\alpha = \{x \in X \mid \alpha = (\alpha \cap x)^{\perp\perp}\}$.

Proof. If $x \in X$ and $\alpha = (\alpha \cap x)^{\perp\perp}$, then for $p = \alpha \cap x$,

$$p \subset x \quad \therefore\ x \in U_p \quad \text{and} \quad \alpha = p^{\perp\perp}.$$
$$\therefore \quad x \in E\alpha.$$

By Lemma 4.12, $p^{\perp\perp} = (p^{\perp\perp} \cap x)^{\perp\perp}$ for $p \subset x$. Hence

$$\begin{aligned} x \in E\alpha &\implies \exists p \subset x(p^{\perp\perp} = \alpha) \\ &\implies (\alpha \cap x)^{\perp\perp} = (p^{\perp\perp} \cap x)^{\perp\perp} = p^{\perp\perp} = \alpha. \end{aligned}$$

□

Definition 4.7. For $\alpha \in \mathcal{L}^{\perp\perp}(Q)$, we define a function f_α on $E\alpha$ by

$$f_\alpha(x) \overset{\text{def}}{=} \bigvee\nolimits^{x^*}(\alpha \cap x) \in x^* \quad \text{for } x \in E\alpha.$$

Theorem 4.17. *If $x \in E\alpha \cap E\beta$, then*

$$f_\alpha(x) \leq f_\beta(x) \iff \alpha \leq \beta.$$

Proof.

$$\begin{aligned} \alpha \leq \beta &\implies \bigvee\nolimits^{x^*}(\alpha \cap x) \leq \bigvee\nolimits^{x^*}(\beta \cap x) \\ &\implies \alpha \cap x \subset \beta \cap x \quad \text{by Lemma 4.14} \\ &\implies \alpha = (\alpha \cap x)^{\perp\perp} \leq (\beta \cap x)^{\perp\perp} = \beta \\ \therefore \quad \alpha \leq \beta &\iff \bigvee\nolimits^{x^*}(\alpha \cap x) \leq \bigvee\nolimits^{x^*}(\beta \cap x). \end{aligned}$$

Therefore,

$$f_\alpha(x) \leq f_\beta(x) \iff \bigvee\nolimits^{x^*}(\alpha \cap x) \leq \bigvee\nolimits^{x^*}(\beta \cap x) \iff \alpha \leq \beta.$$

□

LEMMA 4.18. *If $a \in x \in X$, then*

$$f_{\{a\}^{\perp\perp}}(x) = a \quad \textit{and} \quad f_{\{a\}^{\perp}}(x) = a^{\perp}.$$

Proof. If $a \in x \in X$, then $a^{\perp} \in x$. $\therefore$ $\{a^{\perp}\} \subset x$, and by Lemma 4.5

$$\{a\}^{\perp} = \{a^{\perp}\}^{\perp\perp} = (\{a^{\perp}\}^{\perp\perp} \cap x)^{\perp\perp}.$$

$$\therefore \quad a \in x \Longrightarrow x \in E\{a\}^{\perp\perp} \quad \text{and} \quad x \in E\{a\}^{\perp}.$$

$$f_{\{a\}^{\perp}}(x) = \bigvee\nolimits^{x^*}(\{a\}^{\perp} \cap x) = a^{\perp} \quad \text{and} \quad f_{\{a\}^{\perp\perp}}(x) = \bigvee\nolimits^{x^*}(\{a\}^{\perp\perp} \cap x) = a.$$

□

LEMMA 4.19. *If $x \in \bigcap_i E\alpha_i$, then $x \in E\bigvee_i \alpha_i$ and*

$$f_{\bigvee_i \alpha_i}(x) = \bigvee\nolimits^{x^*} f_{\alpha_i}(x).$$

Proof. If $x \in \bigcap_i E\alpha_i$, then

$$\bigcup_i \alpha_i = \bigcup_i (\alpha_i \cap x)^{\perp\perp} \leq \big(\bigcup_i (\alpha_i \cap x)\big)^{\perp\perp}$$

$$\therefore \quad \bigvee_i \alpha_i \leq \big(\bigcup_i (\alpha_i \cap x)\big)^{\perp\perp} \leq \bigvee_i \alpha_i \quad \therefore \quad \bigvee_i \alpha_i = \big(\bigcup_i (\alpha_i \cap x)\big)^{\perp\perp}.$$

Since $\bigcup_i (\alpha_i \cap x) \subset x$, we have $x \in E\bigvee_i \alpha_i$. □

DEFINITION 4.8. $F(x) \stackrel{\text{def}}{=} \{f_\alpha(x) \mid x \in E\alpha\}$, then $F(x) = x^*$.

DEFINITION 4.9. $F(U) \stackrel{\text{def}}{=} \{f_\alpha \restriction U \mid U \subset E\alpha\}$.

DEFINITION 4.10. If $U, V \in \mathcal{O}(X)$ and $V \subset U$, then

$$r_{V,U} : F(U) \to F(V)$$

is defined by $r_{V,U}(f) = f \restriction V$ for $f \in F(U)$, where $f \restriction V$ is the restriction of $f \in F(U)$ on V.

COROLLARY 4.20. *If $U, V \in \mathcal{O}(X)$ and $V \subset U$, then*

$$F(U) \restriction V = \{r_{V,U}(f) \mid f \in F(U)\} \subset F(V).$$

Obviously, $r_{V,U} : F(U) \to F(V)$ is a homomorphism. Hence, $\langle F, r \rangle$ is a presheaf.

THEOREM 4.21. *$\langle F, r \rangle$ is a sheaf of complete Boolean algebra.*

Proof. Assume that $\{U_i\}_{i \in I} \subset \mathcal{O}(X)$, $U = \bigcup_{i \in I} U_i$ and

$$\forall i \in I\big(f_i \restriction U_i \in F(U_i)\big) \wedge \forall i, j \in I\big(r_{U_i \cap U_j, U_i}(f_i \restriction U_i) = r_{U_i \cap U_j, U_j}(f_i \restriction U_j)\big).$$

Then for each $x \in U$ there exists $i \in I$ such that $x \in U_i$. Let $f(x) = f_i(x)$.

$$(f \restriction U_i)(x) = f_{\alpha_i}(x) \quad \text{for } x \in U_i.$$

It follows that $f \restriction U_i = f_i \restriction U_i$, so $f \restriction U$ is the desired function. □

COROLLARY 4.22. *Each* $\alpha \in \mathcal{L}^{\perp\perp}(Q)$ *is represented by a* **cross-section** $\Gamma(E\alpha, f_\alpha)$ *of the sheaf* $\langle F, r\rangle$*:*

$$\Gamma(E\alpha, f_\alpha) = \{\langle x, f_\alpha(x)\rangle \mid x \in E\alpha\}.$$

Proof. If $\Gamma(E\alpha, f_\alpha) = \Gamma(E\beta, f_\beta)$, then $E\alpha = E\beta$ and $\forall x \in E\alpha(f_\alpha(x) = f_\beta(x))$.

$$\therefore \quad \bigvee^{x^*}(\alpha \cap x) = \bigvee^{x^*}(\beta \cap x)$$

$$\therefore \quad \alpha = (\alpha \cap x)^{\perp\perp} = (\beta \cap x)^{\perp\perp} = \beta.$$

□

4.5 Interpretation

DEFINITION 4.11. For $|\varphi| \in Q$,

$$[\![\varphi]\!] \overset{\text{def}}{=} \{|\varphi|\}^{\perp\perp}$$

$$E[\![\varphi]\!] \overset{\text{def}}{=} \bigcup\{U_p \in \mathcal{O}(X) \mid \exists x \in X\big(p \subset x \wedge ([\![\varphi]\!] = p^{\perp\perp})\big)\},$$

where $U_p = \{x \in X \mid p \subset x\}$.

By Proposition 4.1 and Corollary 4.6 , we have

LEMMA 4.23. (1) $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$

(2) $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$

(3) $[\![\forall x \varphi(x)]\!] = \bigwedge\{[\![\varphi(a)]\!] \mid a \in FV\}$

(4) $[\![\exists x \varphi(x)]\!] = \bigvee\{[\![\varphi(a)]\!] \mid a \in FV\}$

(5) $[\![\varphi^\perp]\!] = [\![\varphi]\!]^\perp$

(6) $[\![\Box\varphi]\!] = \Box[\![\varphi]\!]$

$\langle E[\![\varphi]\!], f_{[\![\varphi]\!]}\rangle$ is the truth value of φ. That is,

$$\mathcal{M} = \langle \mathcal{F}, D, I\rangle, \quad \text{where}$$

$$\mathcal{F} = \{\langle E[\![\varphi]\!], f_{[\![\varphi]\!]}\rangle \mid \varphi \text{ is a formula of } \mathrm{QL}_\Box\}, \quad D = FV, \quad I(p) = p.$$

4.6 Proof of sheaf completeness

THEOREM 4.24 (Sheaf completeness of $\mathrm{QL}_\Box$)**.** *If* φ *is a formula of* $\mathrm{QL}_\Box$*, then*

$$\models \varphi \quad \Longrightarrow \quad \mathrm{QL}_\Box \vdash \varphi.$$

Proof. By Proposition 4.1, $|\varphi_0| \le |p| \lneqq |p \vee q|$ for the non-provable formula φ_0 and the propositional constants p, q. Hence there exists $x \in X$ such that $\{|\varphi_0|, |p|, |p \vee q|\} \subset x$.

$$\therefore \quad |\varphi_0| \neq 1. \qquad \therefore \quad f_{[\![\varphi_0]\!]} \neq 1.$$

Therefore, $\langle \mathcal{M}, v\rangle \not\models \varphi_0$. Hence $\not\models \varphi_0$. □

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