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Some Properties of Codes with Infinite Deciphering Delay

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Abstract: In 2013, Tommi Lehtinen and Alexander Okhotin proved that if X is a code, then it has infinite deciphering delay if and only if there exist $x, y, z \in A^+$ with $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. In this paper, we give a sufficient and necessary condition for codes with infinite deciphering delay. Then, we construct two kinds of three-element codes with infinite deciphering delay.

Keywords: Infinite Deciphering Delay, Prefix Graph, Construction, Injective Morphism

1 Introduction

A very important property required for codes is that decoding a transmitted message is possible before its complete reception. This quality is satisfied in particular by codes with a finite deciphering delay. Codes with finite deciphering delay have important application in the field of information theory and coding theory (see [1,4-5]). In 2001, Jean Néraud and Carla Selmi proved that if X is a non-complete code with a finite deciphering delay, then there exists an uncompletable word w of length $O(m^2d^2)$, where d stands for the delay and m stands for the length of the longest words in X (see [6]). In 2014, Lila Kari and Stavro Knostantindis considered the problem of deciding maximality of a regular language with respect to deciphering delay 1 and transducer-based property, such as suffix code, overlap-free language and error-detection properties (see [7]).

However, researches on codes with infinite deciphering delay help us to know them from another aspect, and they have important application in information theory. For example, the characterization of codes with infinite deciphering delay is used to construct examples of languages witnessing the non-closure under certain morphism (see [2]). In this paper, we give some properties of codes with infinite deciphering delay. First, we show that if X is a code, then it has infinite deciphering delay if and only if there exist three paths in the prefix graph of X. Second, we prove that codes with infinite deciphering delay preserve injective morphism. Finally, we give two sufficient conditions on three-element codes with infinite deciphering delay. The proofs lead explicit constructions of these three-element codes.

2 Preliminaries

Let A be a finite set, which is called an alphabet. An element $a \in A$ is called a letter. A word w on the alphabet A is a finite sequence of letters in A, that is $w = a_1 a_2 \ldots a_n$ where $a_i \in A$ and $i = 1, \ldots, n$. The set of all words on the alphabet A is denoted by A^* and is equipped with the associative operation defined by $(a_1 a_2 \ldots a_n) \cdot (b_1 b_2 \ldots b_m) = a_1 a_2 \ldots a_n b_1 b_2 \ldots b_m$. The empty sequence is called the empty word and is denote by e. It is the neutral element for concatenation. Then A^* is a free monoid generated by A. Let $A^+ = A^* \setminus \{e\}$. The length $|w|$ of a word $w = a_1 a_2 \ldots a_n$ with $a_i \in A, i = 1, \ldots, n$ is the number n of letters in w. So $|e| = 0$. A nonempty word w is a *primitive word*, if w is not a power of any other nonempty word. Let Q be the set of all primitive words over A. Let A, B be alphabets, and $Z \subseteq A^*$, $Y \subseteq B^*$ be two codes. Then the codes Y and Z are *composable* if there is a bijection from B onto Z. If b is such a bijection, then Y and Z are called composable through b. Then b defines a morphism from B^* into A^* which is injective since Z is a code. The set $X = b(Y) \subseteq Z^* \subseteq A^*$ is obtained by *composition* of Y and Z (by means of b). We denote by $X = Y \circ_b Z$ or $X = Y \circ Z$, when the context permits it.

Let X be a finite set of words over alphabet A. We define a graph G_X of X, which is called the *prefix graph* of X as follows. The vertices of G_X are the nonempty prefixes of words in X, and there is an edge from s to t if and only if one of the two following situation occurs: either $st \in X$ or $sx = t$ for some $x \in X$. Edges of the first type are called *crossing*, and edges of the second type are called *extending*. A crossing edge (s, t) is labeled with the word t, and an extending edge (s, t) with $sx = t$ is labeled with x. Two factorizations $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$ of a word are disjoint if $x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_m$ and $x_1 x_2 \ldots x_i \neq y_1 y_2 \ldots y_j$ for any $1 \le i < n,\ 1 \le j < m$. A nonempty subset X of A^* is said to have *finite verbal deciphering delay* if there exists an integer $d \ge 0$ such that for any $x, x', y' \in X$ and $y \in X^d$, $xy \le_p x'y'$ implies $x = x'$. If no such integer exists, the set X has *infinite deciphering delay*.

Lemma 2.1[2] Let $X \subseteq A^*$ be a code. Then it has infinite deciphering delay if and only if there exist $x, y, z \in A^+$ with $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$.

Lemma 2.2[1] There is a path of length $n\ (\geq 1)$ from s to t in the prefix graph of X if and only if there exist $x_1, \ldots, x_k, y_1, \ldots, y_l$ in X such that $s y_1 \ldots y_l t = x_1 \ldots x_k$ or $s y_1 \ldots y_l = x_1 \ldots x_k t$ are disjoint factorizations with $k + l = n$, where s is a prefix of x_1 or s is a prefix of t if $k = 0$. The label of the path is $y_1 \ldots y_l t$ in the first case and $y_1 \ldots y_l$ in the second case. The first (second) case occurs if and only if the path contains an odd (even) number of crossing edges.

Lemma 2.3[1] The nonempty set $X \subseteq A^+$ is a code if and only if none of the set U_n contains the empty word, where $U_1 = (X^{-1}X) \setminus \{e\}$ and $U_{n+1} = X^{-1}U_n \cup U_n^{-1}X$ when $n \geq 1$.

Lemma 2.4[1] Let $\alpha : A^* \to C^*$ be an injective morphism. If X is a code over alphabet A, then $\alpha(X)$ is a code over C. If Y is a code over C, then $\alpha^{-1}(Y)$ is a code over A.

Lemma 2.5 [3] If $uv = vz$ where $u, v, z \in A^*$ and $u \neq e$, then $u = xy, v = (xy)^k x, z = yx$ for some $x, y \in A^*$ and $k \geq 0$.

Lemma 2.6[3] If $uv = vu$ where $u \neq e, v \neq e$, then u and v are powers of a common word.

3 Infinite deciphering delay

In this section, we give a sufficient and necessary condition for codes with infinite deciphering delay, then construction methods are given for three-element codes with infinite deciphering delay. We first prove a lemma which will be used in the later results.

Lemma 3.1 Let $X \subseteq A^*$ be a code. If $sy_1 \ldots y_n t = x_1 \ldots x_m$, where $n \geq 0, m \geq 1$, $x_1, \ldots, x_m, y_1, \ldots, y_n \in X, s <_p x_1$, $t \in A^*$ and $y_1 \ldots y_n t \notin X^*$. Then two factorizations $(s, y_1, \ldots, y_n, t)$ and $(x_1, \ldots, x_m)$ are disjoint.

Proof: We prove the result by induction on $n + m$. If $n + m = 1$, then $st = x_1$. Since $y_1 \ldots y_n t \notin X^*$, then $t \neq e$. Thus (s, t) and (x_1) are disjoint. Assume the result holds for $n + m \leq k - 1$. If $n + m = k$, let $sy_1 \ldots y_n t = x_1 \ldots x_m$. Since $y_1 \ldots y_n t \notin X^*$ then $|t| \neq |x_m|$. Then one of the following two cases holds.

(1) If $|t| > |x_m|$, there exists $t_1 \in A^+$ such that $t_1 x_m = t$. So $sy_1 \ldots y_n t_1 = x_1 \ldots x_{m-1}$, and $y_1 \ldots y_n t_1 \notin X^*$. Now $m + n - 1 \leq k - 1$. By the induction assumption, we know two factorizations $(s, y_1, \ldots, y_n, t)$ and $(x_1, \ldots, x_{m-1})$ are disjoint. Thus two factorizations $(s, y_1, \ldots, y_n, t)$ and $(x_1, \ldots, x_m)$ are disjoint because x_m is a proper suffix of t.

(2) If $|t| < |x_m|$, then $sy_1 \ldots y_{i-1} w = x_1 \ldots x_{m-1}$, $y_i = wv$ and $vy_{i+1} \ldots y_n t = x_m$ for some $w, v \in A^*$ and $1 \leq i \leq n$. In fact $w, v \neq e$. Suppose $w = e$. Then $y_i y_{i+1} \ldots y_n t = x_m \in X$.

Hence $y_1 \ldots y_i y_{i+1} \ldots y_n t = y_1 \ldots y_i x_m \in X^*$, which is a contraction. By the same way, we know $v \neq e$. Suppose that $y_1 \ldots y_{i-1} w \in X^*$, then $y_1 \ldots y_{i-1} w v y_{i+1} \ldots y_n t = y_1 \ldots y_{i-1} w x_m$, which is a contraction. Hence $y_1 \ldots y_{i-1} w \notin X^*$. Since $s y_1 \ldots y_{i-1} w = x_1 \ldots x_{m-1}$, by the induction assumption, two factorizations $(s, y_1, \ldots, y_{i-1}, w)$ and $(x_1, \ldots, x_{m-1})$ are disjoint. Thus two factorizations $(s, y_1, \ldots, y_n, t)$ and $(x_1, \ldots, x_m)$ are disjoint because $v y_{i+1} \ldots y_n t = x_m$. □

Theorem 3.2 Let $X \subseteq A^+$ be a code. Then it has infinite deciphering delay if and only if there exist three paths $(\tilde{z}, y')$, $(y', \tilde{z})$ and $(\tilde{x}, y')$ in the prefix graph of X, where $\tilde{x}, y', \tilde{z} \in A^+$ and $X^* \tilde{x} \cap X^* \neq \varnothing$. The labels of these three paths are y, z and y, respectively, and $y, z \notin X^*$. Each path contains an odd number of crossing edges.

Proof: ($\Rightarrow$) Let X is a code with infinite deciphering delay. By lemma 2.1, then $x, xy, yz, zy \in X^*$ for some $x, y, z \in A^+$ and $y, z \notin X^*$.

Since $yz \in X^*$ and $y, z \notin X^*$, then $y = u_1 \ldots u_n y'$, $z = z' v_1 \ldots v_m$ and $y'z' \in X$ for some $u_1, \ldots, u_n, v_1, \ldots, v_m \in X$, $y', z' \in A^+$. Hence $y' <_s y$ and $z' <_p z$. Since $zy \in X^*$, then $y = \tilde{y} \tilde{u}_1 \ldots \tilde{u}_p$, $z = \tilde{v}_1 \ldots \tilde{v}_q \tilde{z}$ and $\tilde{z}\tilde{y} \in X$ for some $\tilde{u}_1, \ldots, \tilde{u}_p, \tilde{v}_1, \ldots, \tilde{v}_q \in X$, $\tilde{y}, \tilde{z} \in A^+$. So $\tilde{z} <_s z$, $\tilde{y} <_p y$. Since $xy \in X^*$, then $y = y'' v'_1 \ldots v'_r$, $x = u'_1 \ldots u'_l \tilde{x}$ and $\tilde{x} y'' \in X$ for some $u'_1, \ldots, u'_l, v'_1, \ldots, v'_r \in X$, $y'', \tilde{x} \in A^+$. Hence $y'' <_p y$, $\tilde{x} <_s x$ and $X^* \tilde{x} \cap X^* \neq \varnothing$.

(1) Since $y = u_1 \ldots u_n y' = \tilde{y} \tilde{u}_1 \ldots \tilde{u}_p$, then $\tilde{z} u_1 \ldots u_n y' = \tilde{z}\tilde{y} \tilde{u}_1 \ldots \tilde{u}_p$. Let $s_1 = \tilde{z}$, $t_1 = y', u_i = y_i^{(1)}, i = 1, \ldots, n$, $x_1^{(1)} = \tilde{z}\tilde{y}$ and $x_{j+1}^{(1)} = \tilde{u}_j, j = 1, \ldots, p$. Hence we have $s_1 y_1^{(1)} \ldots y_{n_1}^{(1)} t_1 = x_1^{(1)} \ldots x_{m_1}^{(1)}$. By lemma 3.1, two factorizations $(s_1, y_1^{(1)}, \ldots, y_{n_1}^{(1)}, t_1)$ and $(x_1^{(1)}, \ldots, x_{m_1}^{(1)})$ are disjoint, where $n_1 = n, m_1 = p + 1$. By lemma 2.2, there is a path of length $n_1 + m_1$ from $\tilde{z}$ to y' in the prefix graph of X. The label of the path is $y_1^{(1)} \ldots y_{n_1}^{(1)} y'$. Since $y = y_1^{(1)} \ldots y_{n_1}^{(1)} y'$, then the label of the path is y. The path contains an odd number of crossing edges.

(2) Since $z = \tilde{v}_1 \ldots \tilde{v}_q \tilde{z} = z' v_1 \ldots v_m$, then $y' \tilde{v}_1 \ldots \tilde{v}_q \tilde{z} = y'z' v_1 \ldots v_m$. Let $s_2 = y'$, $t_2 = \tilde{z}, \tilde{v}_i = y_i^{(2)}, i = 1, \ldots, q$, $x_1^{(2)} = y'z'$ and $x_{j+1}^{(2)} = v_j, j = 1, \ldots, m$. So $s_2 y_1^{(2)} \ldots y_{n_2}^{(2)} t_2 = x_1^{(2)} \ldots x_{m_2}^{(2)}$. By lemma 3.1, two factorizations $(s_2, y_1^{(2)}, \ldots, y_{n_2}^{(2)}, t_2)$ and $(x_1^{(2)}, \ldots, x_{m_2}^{(2)})$ are disjoint, where $n_2 = q, m_2 = m + 1$. By lemma 2.2, there exist a path $(y', \tilde{z})$ in the prefix graph of X. The label of the path is $y_1^{(2)} \ldots y_{n_2}^{(2)} \tilde{z}$. Since $z = y_1^{(2)} \ldots y_{n_2}^{(2)} \tilde{z}$, then the label of the path is z. The path contains an odd number of crossing edges.

(3) Since $y = u_1 \ldots u_n y' = y'' v'_1 \ldots v'_r$ then $\tilde{x} u_1 \ldots u_n y' = \tilde{x} y'' v'_1 \ldots v'_r$. Let $s_3 = \tilde{x}$, $t_3 = y', u_i = y_i^{(3)}, i = 1, \ldots, n$, $x_1^{(3)} = \tilde{x} y''$ and $x_{j+1}^{(3)} = v'_j, j = 1, \ldots, r$. Thus we have $s_3 y_1^{(3)} \ldots y_{n_3}^{(3)} t_3 = x_1^{(3)} \ldots x_{m_3}^{(3)}$. By lemma 3.1, two factorizations $(s_3, y_1^{(3)}, \ldots, y_{n_3}^{(3)}, t_3)$ and

$(x_1^{(3)}, \mathbf{K}, x_{m_3}^{(3)})$ are disjoint, where $n_2 = q, m_2 = m+1$. $n_3 = n, m_3 = r+1$. There exists a path $(\tilde{x}, y')$ in the prefix graph of X. The label of the path is $y_1^{(3)} \mathbf{K}\, y_{n_3}^{(3)} y'$. Since $y = y_1^{(3)} \mathbf{K}\, y_{n_3}^{(3)} y'$, then the label of the path is y. The path contains an odd number of crossing edges.

($\Leftarrow$) Suppose that there exist three paths $(\tilde{z}, y')$, $(y', \tilde{z})$ and $(\tilde{x}, y')$ in the prefix graph of X, which satisfy the conditions given as above. By lemma 2.2, we have $\tilde{z} u_1 \mathbf{K}\, u_n y' = \tilde{u}_1 \mathbf{K}\, \tilde{u}_p$, $y' \tilde{v}_1 \mathbf{K}\ \tilde{v}_q \tilde{z} = v_1 \mathbf{K}\ v_m$ and $\tilde{x} u_1 \mathbf{K}\, u_n y' = v_1' \mathbf{K}\, v_r'$ for some $u_1, \mathbf{K}, u_n, v_1, \mathbf{K}, v_m$, $\tilde{u}_1, \mathbf{K}, \tilde{u}_p, \tilde{v}_1, \mathbf{K}, \tilde{v}_q$, $v_1', \mathbf{K}, v_r' \in X$. Since $X^* \tilde{x}\ \mathbf{I}\ X^* \neq \emptyset$, let $x \in X^* \tilde{x}\ \mathbf{I}\ X^*$. Then there exist $x_1, \mathbf{K}, x_t \in X$ and $t \geq 0$ such that $x = x_1 \mathbf{K}\, x_t \tilde{x}$. Since the label of these paths are y, z and y, we have $y = u_1 \mathbf{K}\, u_n y'$ and $z = \tilde{v}_1 \mathbf{K}\, \tilde{v}_q \tilde{z}$. So $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. Thus X is a code with infinite deciphering delay by lemma 2.1. □

Example 3.3 Let $A = \{a, b\}$ and $X = \{ababa,\ \ baaba,\ \ ababaab\}$. On one hand, the prefix graph of X is given in figure 1. By figure 1, X is a code. Let $\tilde{z} = ba, y' = bab$ and $\tilde{x} = ab$, there exist paths $(ba, aba), (aba, ba)$ and (ab, aba) in G_X. The labels of the three paths are aba, ba, aba. Then X is a code with infinite deciphering delay by theorem 3.2. On the other hand, we denote $y = aba, z = ba$ and $x = ababaab$. Therefore, $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. By lemma 2.1, X is a code with infinite deciphering delay.

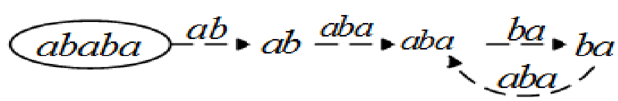

Figure 1. The prefix graph of X: G_X

Proposition 3.4 Let $a : A^* \to C^*$ be an injective morphism. If X is a code with infinite deciphering delay over A, then $a(X)$ is a code with infinite deciphering delay over C.

Proof: By lemma 2.1, there exist $x, y, z \in A^+$ with $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. By lemma 2.4, $a(X)$ is a code. Since a is injective, then $a(x)$, $a(y)$, $a(z) \in C^+$. Since a is a morphism, then $a(x)a(y) = a(xy) \in a(X)^*$, $a(y)a(z) = a(yz) \in a(X)^*$ and $a(z)a(y) \in a(X)^*$. Hence we have $a(x)$, $a(x)a(y)$, $a(y)a(z)$, $a(z)a(y) \in a(X)^*$. Suppose $a(y) \in a(X)^*$. Then there exist $y_1, \mathbf{K}, y_n \in X$ such that $a(y) = a(y_1) \mathbf{K}\, a(y_n) = a(y_1 \mathbf{K}\, y_n)$. Since a is an injective morphism, then $y = y_1 \mathbf{K}\, y_n \in X^*$, which is a contradiction. Similarly, we have $a(z) \notin a(X)^*$. By lemma 2.1, $a(X)$ is a code with infinite deciphering delay.□

Corollary 3.5 Let $Z \subseteq A^*$ and $Y \subseteq B^*$ be composable codes and $X = Y \mathbf{o}_b Z$. If Y is a code with infinite deciphering delay, then $X = b(Y)$ which is a code with infinite deciphering delay.

4 Three-elements codes with infinite deciphering delay

We know that every two-element code is a code with finite deciphering delay (see [1]). In the following, we present two sufficient conditions for three-elements codes with infinite deciphering delay.

Proposition 4.1 Let $y, z \in A^+$ and $yz \neq zy$. If one of the following conditions holds:

(1) $\tilde{x}y\tilde{x} = zy$ and $|z| < |\tilde{x}y|$;

(2) $\tilde{x} = zy$,

then $X = \{\tilde{x}y, \quad yz, \quad zy\}$ is a code with infinite deciphering delay.

Proof: (1) If $\tilde{x}y\tilde{x} = zy$ and $|z| < |\tilde{x}y|$, then $z = \tilde{x}y_1$, $y = y_2\tilde{x}$ and $y = y_1y_2$ for some $y_1, y_2 \in A^+$. Since $y = y_1y_2 = y_2\tilde{x}$, by lemma 2.5, then $y_1 = uv$, $y_2 = (uv)^k u$, $\tilde{x} = vu$ for some $u, v \in A^*$ and $k \geq 0$. Hence $y = y_1y_2 = (uv)^{k+1}u$ and $z = \tilde{x}y_1 = vu^2v$. Since y and z are not powers of a common word, then $u, v \in A^+$. Suppose $uv = vu$, then $u^2v = vuv$. This implies $yz = (uv)^{k+2}u^2v = (uv)^{k+2}uvu = vu^2v(uv)^k u = zy$, which contracts $yz \neq zy$. So $X = \{\tilde{x}y, \ yz, \ zy\} = \{vu(uv)^{k+1}u, \ (uv)^{k+2}u^2v, \ vu(uv)^{k+2}u\}$. Then $U_1 = \{vu\}$, $U_2 = \{(uv)^{k+1}u, \quad (uv)^{k+2}u\}$, $U_3 = \{vu^2v, \quad uv\}$, $U_4 = \{(uv)^k u, \ (uv)^{k+1}u, \quad (uv)^{k+1}u^2v\}$. For any $n \geq 3$, we have $U_{2n+1} = U_5 = \{vu^2v, vuvu^2v\}$ and $U_{2n} = U_6 = \{(uv)^{k+1}u, (uv)^k u\}$. By lemma 2.3, X is a code. Let $x = \tilde{x}y\tilde{x} = zy \in X^*$. Then $xy = (\tilde{x}y)^2 \in X^*$. Hence $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. Thus X is a code with infinite deciphering delay by lemma 2.2.

(2) If $\tilde{x} = zy$, then $X = \{\tilde{x}y, \quad yz, \quad zy\} = \{zy^2, \quad yz, \quad zy\}$. Since y and z are not powers of a common word, then $U_1 = \{y\}$, $U_2 = \{z\}$, $U_3 = \{y, y^2\}$. For any $n \geq 1$, we have $U_{2n+1} = U_3$ and $U_{2n} = U_2$. By lemma 2.3, X is a code. Let $x = zy$. Then $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. Thus X has infinite deciphering delay. □

Example 4.2 Let $\tilde{x}_1 = ba$, $y_1 = aba$, $z_1 = ba^2b$. Then $\tilde{x}_1 y_1 \tilde{x}_1 = ba^2baba = z_1y_1$. Thus $X_1 = \{\tilde{x}_1y_1, \quad y_1z_1, \quad z_1y_1\} = \{baaba, \quad ababaab, \quad baababa\}$, which is a code with infinite deciphering delay. Let $\tilde{x}_2 = ba$, $y_2 = a$, $z_2 = b$. Then $\tilde{x}_2 = z_2y_2$. Hence we have $X_2 = \{\tilde{x}_2y_2, \quad y_2z_2, \quad z_2y_2\} = \{baa, \quad ba, \quad ab\}$ is a code with infinite deciphering delay. The prefix graphs of X_1 and X_2 are given in figure 2.

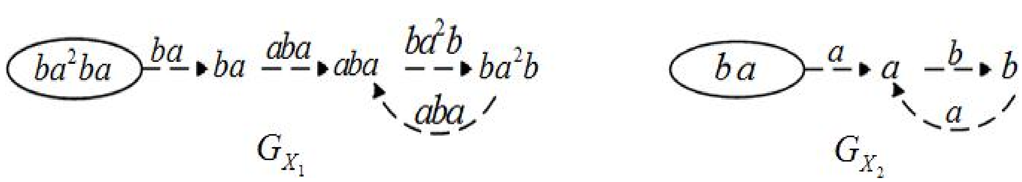

Figure 2. The prefix graphs of X_1 and X_2

Theorem 4.3 Let $y, z \in A^+$ and $yz = zy$. If $x_1\tilde{x} = \tilde{x}y$ or $\tilde{x}y = yz$ where f is the primitive root of y and z and $x_1 \not\leq_p f$, $f \not\leq_p x_1$, then $X = \{x_1, \quad x_1\tilde{x}, \quad yz\}$ is a code with infinite deciphering delay.

Proof: (1) If $x_1\tilde{x} = \tilde{x}y$, by lemma 2.5, then $x_1 = uv$, $\tilde{x} = (uv)^{k_1}u$, $y = vu$ for some $u, v \in A^*$ and $k_1 \geq 0$. Since x_1 and f are incomparable for the prefix order, then $uv \neq vu$ and $u, v \in A^+$. Let $z = f^t$ for some $t \geq 1$. Hence $X = \{uv, \quad (uv)^{k_1+1}u, \quad vuf^t\}$. Then $U_1 = \{(uv)^{k_1}u\}$, $U_2 = \{vu\}, U_3 = \{f^t\}$. Since $y = vu = f^i$, then $U_4 = \{vu\}$. For any $n \geq 1$, we have $U_{2n+1} = U_3$ and $U_{2n} = U_2$. By lemma 2.3, X is a code. Let $x = x_1\tilde{x} = \tilde{x}y$. Then $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. Thus X is a code with infinite deciphering delay.

(2) If $\tilde{x}y = yz$, then there exist $u_1, v_1 \in A^*$ and $k_2 \geq 0$ such that $\tilde{x} = u_1v_1$, $y = (u_1v_1)^{k_2}u_1$ and $z = v_1u_1$. Since y and z are powers of a common word, then $zy = yz$. It implies $(u_1v_1)^{k_2+1}u_1 = v_1u_1(u_1v_1)^{k_2}u_1$. Thus $u_1v_1 = v_1u_1$. We consider the following two cases.

(2-1) If $u_1, v_1 \in A^+$, by lemma 2.5, u_1 and v_1 are powers of a common word. Therefore, $X = \{x_1, \quad x_1u_1v_1, \quad (u_1v_1)^{k_2+1}u_1\}$. Let $y = (u_1v_1)^{k_2}u_1 = f^i$ for some $i \geq 1$. Then $u_1 = f^r, v_1 = f^s$ for some $r, s \geq 1$. Hence $X = \{x_1, \quad x_1f^{r+s}, \quad f^{(r+s)(k_2+1)+r}\}$. Since x_1 and f are incomparable under the prefix order, then $U_1 = \{f^{r+s}\}$ and $U_2 = \{f^{(r+s)k_2+r}\}$. For any $n \geq 1$, we have $U_{2n+1} = U_1$ and $U_{2n} = U_2$. By lemma 2.3, X is a code. Let $x = x_1\tilde{x} = \tilde{x}y$. Then $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. Thus X has infinite deciphering delay.

(2-2) If $u_1 = e$ or $v_1 = e$, without loss generality, we let $u_1 = e$. Then $\tilde{x} = v_1$, $y = v_1^{k_2}$ and $z = v_1$. Since $y, z \neq e$, then $v_1 \neq e$ and $k_2 \geq 1$. Hence $X = \{x_1, \quad x_1v_1, \quad v_1^{k_2+1}\}$. Since x_1 and f are incomparable for the prefix order, then x_1 and v_1 are incomparable for the prefix order. Therefore, $U_1 = \{v_1\}$ and $U_2 = \{v_1^{k_2}\}$. For any $n \geq 1$, we have $U_{2n+1} = U_1$ and $U_{2n} = U_2$. By lemma 2.3, X is a code. Let $x = x_1v_1$. Then $x, xy, yz, zy \in X^*$ and $y, z \notin X^*$. Thus X has infinite deciphering delay. □

Example 4.4 Let $\tilde{x} = a$, $y = ba$, $z = baba$, $x_1 = ab$. Then $yz = zy$ and $x_1\tilde{x} = \tilde{x}y$. Then $X_3 = \{x_1, \quad x_1\tilde{x}, \quad yz\} = \{ab, \quad aba, \quad (ba)^3\}$, which is a code with infinite deciphering delay. Let $\tilde{x}' = (ab)^3$, $y' = ab$, $z' = (ab)^3$, $x_1' = aa$. Then $y'z' = z'y'$ and $\tilde{x}'y' = y'z'$. Therefore, $X_4 = \{x_1', \quad x_1'\tilde{x}', \quad y'z'\} = \{aa, \quad a^2(ab)^3, \quad (ab)^4\}$, which is a code with infinite deciphering delay.

Example 4.5 Let $B = \{a, b, c\}$ and $Y = \{ab, \quad abc, \quad cc\}$. Denote $x_1 = ab$, $\tilde{x} = c$, $y = z = c$. Then $\tilde{x}y = yz = c^2$. By theorem 4.3, Y is a code with infinite deciphering delay. Let $A = \{0, 1\}$ and $Z = \{01010, \quad 10010, \quad 0101001\}$. We define $b(a) = 01010, \quad b(b) = 10010, \quad b(c) = 0101001$. Then

$$X = Y \circ Z = \{0101010010, \quad 01010100100101001, \quad 01010010101001\}.$$

By corollary 3.5, X is a code with infinite deciphering delay. In fact, by example 3.3, Z is a code with infinite deciphering delay.

5 Conclusion

By example 4.5, we know Z is a three-elements code with infinite deciphering delay, but it doesn't satisfies the conditions in theorem 4.1 and 4.3. In the future, we want to investigate the other constructions of three-elements codes with infinite deciphering delay.

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