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The Operations of Two Maximal Left Singular Languages[1]

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Abstract: In this paper, we obtain the following results on maximal left singular languages. The production of two maximal left singular languages is not a maximal left singular language. The union of two maximal left singular languages is a maximal left singular language if and only if their left singular words are the same word. The intersection of two maximal left singular languages is always a non-empty set.
Keywords: Maximal Left Singular Language, Left Singular Word, Production, Union, Intersection

1 Introduction and Preliminaries

Formal language theory studies the purely syntactical aspects of languages and their internal structural patterns. It sprang out of linguistics, as a way of understanding the syntactic regularities of natural languages and can be widely used in computer science, information theory, control theory etc. (see [14,16-22]). Prefix codes playing important role in information theory and computer science (see [14-17]). Both prefix codes and left singular languages are left cancellative

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languages (see [1,2,8]). In theory of semigroups, the left cancellative subsemigroup of a semigroup is very important. In this paper, we consider the closure properties under production, union and intersection of two maximal left singular languages.

Let X be a nonempty finite set which is called an alphabet. Any finite string over X is called a word. For instance, $w = bbbabaaa$ is a word over $\{a, b\}$. The word which contains not any letter is called the empty word, denoted by 1. The set of all words is denoted by X^*, which is a free monoid with concatenation. For instance, the production of two words $x = bb$ and $y = babaaa$ is the word $w = xy = bbabbbaa$. For any word w in X^*, let $\lg(w)$ be the number of letters that occur in w and $\lg(1) = 0$. Then $\lg(w) = 8$ for the former word $w = ababbaaa$. The powers of a word $w \in X^*$ are defined inductively: $w^0 = 1$ and $w^{n+1} = w^n w$ for $n \geq 0$. For instance, if $w = aaaa = a^4$ where $a \in X$, we call it the fourth power of a and its length is 4.

Let $X^+ = X^* \setminus \{1\}$. Let $L \subseteq X^+$ or $L = \{1\}$ is called a language. Let A, B be languages and the production of A, B is defined by $AB = \{xy \mid \text{for any } x \in A, \ y \in B\}$. Let $A \subseteq X^+$ be a language and $l(A) = \{g \in A \mid gx \notin A$ for all $x \in X^+$ and $g = yz$ for all $y \in A$ and $z \in X^*$ implies $z = 1\}$. Every $g \in l(A)$ is called a left singular word in A. If $l(A) \neq \varnothing$ then we call A a left singular language. A left singular language is called a maximal left singular language if for any $x \in X^+ \setminus A$, $A \bigcup \{x\}$ is not a left singular language. If $l(A) = A$, then A is called a prefix code. Let $P(X)$ be the set of all prefix codes over X. Definitions which are used in the paper but not stated here can be found in [2,4,11,14].

Lemma 1.1. [7] Let A be a left singular language who contains only one left singular word x. Then A is a maximal left singular language if only if $\{x, y\}$ is not a prefix code for any $y \in X^+ \setminus A$.

The paper is organized as follows. Some related definitions and preliminaries are presented in section 2. In section 3, we obtain the following results.

(1) If AB is a maximal left singular language, then B must not be a maximal left singular language;

(2) If A, B are maximal left singular languages, then AB must not be a maximal left singular language;

(3) If A, B are maximal left singular languages with different singular word, then $A \bigcup B$ must not be a maximal left singular language;

(4) If A, B are maximal left singular languages, then $A \bigcap B \neq \varnothing$;

(5) If A is a maximal left singular languages, then the image of A must not be a maximal left singular language.

2 The Operations of Maximal Left Singular Languages

If $X = \{a\}$, we can see $\{a^i\}$ and $\{1\}$ are all maximal left singular languages for all $i \geq 1$. Then the production of two maximal left singular languages is also a maximal left singular language. The union of two different maximal left singular languages is not a left singular language, and the intersection of two different maximal left singular languages is an empty set. So in the following, we always let $|X| \geq 2$.

Lemma 2.1. [6] Let $A, B \subseteq X^+$. If AB is a left singular language, then B is a left singular language.

Lemma 2.2. [2] If $g \in l(A)$ and $h \in l(B)$, then $gh \in l(AB)$.

Theorem 2.3. Let $A \subseteq X^+$ be a left singular language. If AB is a maximal left singular language, then B is a maximal left singular language.

Proof. Since AB is a maximal left singular language, then B is a left singular language by lemma 2.1. Suppose B is not a maximal left singular language. Then there exists $x \in X^+ \setminus B$ such that $B \bigcup \{x\}$ is a left singular language. Since A is a left singular language, then $A(B \bigcup \{x\}) = AB \bigcup Ax$ is a left singular language by lemma 2.2. Let $Ax = C_1 \bigcup C_2$ where $C_1 \subseteq AB$ and $C_2 = Ax \setminus AB$. Then $AB \bigcup Ax = AB \bigcup C_2$. For any $w \in C_2$, we have $AB \bigcup \{w\}$ is not a left singular language, because AB is a maximal left singular language. Let $z \in AB$ be the unique left singular word in AB. Then $w = zy_1$ or $z = wy_2$ where $y_1, y_2 \in X^+$. So every word in $AB \bigcup C_2$ is not a left singular word. Hence $AB \bigcup Ax = AB \bigcup C_2$ is not a left singular language. This is a contradiction. Thus B is a maximal left singular language. □

Lemma 2.4. [7] Let A be a left singular language who contains only one left singular word x. Then A is a maximal left singular language if only if $\{x, y\}$ is not a prefix code for any $y \in X^+ \setminus A$.

Let $w \in X^+$. We denote $wX^+ = \{wx \in X^+ \mid \text{for any } x \in X^+\}$ and $wX^- = \{y \in X^* \mid$ there exists $x \in X^+$ such that $w = yx\} = \{y \in X^* \mid y \leq_p w\} \setminus \{w\}$ (see [4]).

Lemma 2.5. [9] Let A be a left singular language and $g \in l(A)$. Then A is a maximal left singular languages if and only if $X^* = gX^+ \bigcup gX^- \bigcup A$.

Theorem 2.6. If $A, B \subseteq X^+$ are two maximal left singular languages, then AB is not a maximal left singular language.

Proof. Since $A, B \subseteq X^+$ are two maximal left singular languages, let $g \in l(A)$ and $h \in l(B)$. Then $gh \in l(AB)$ by lemma 2.2. So AB is a left singular language. Suppose AB is a maximal left singular language. By lemma 2.5, we have $X^* = ghX^+ \bigcup ghX^- \bigcup AB$. If the first letters of g, h are the same, let $g = ax_1$, $h = ax_2$ for some $a \in X$ and $x_1, x_2 \in X^*$. We consider the word b where $b \in X$ and $b \neq a$. Since $b \notin ghX^+, b \notin ghX^-$, then $b \in AB$. But $\lg(b) = 1 < 2 \leq \lg(AB)$. So $b \notin AB$. This is a contradiction. If the first letters of g, h are different, let $g = ax_3, h = bx_4$ for some $a,b \in X$ and $x_3, x_4 \in X^*$. Then $b \notin ghX^+$ and $b \notin ghX^-$. So $b \in AB$. This is a contradiction. Thus AB is not a maximal left singular language. □

Proposition 2.7. Let $A, B \subseteq X^+$ be two maximal left singular languages and $g \in l(A), h \in l(B)$. If first letters of g, h are different, then $A \bigcup B = X^+$.

Proof. Since $A, B \subseteq X^+$ are two maximal left singular languages and $g \in l(A)$, $h \in l(B)$. then $X^* = gX^+ \bigcup gX^- \bigcup A = hX^+ \bigcup hX^- \bigcup B$ by lemma 2.5. Since the first letters of g, h are different, Then $gX^+ \bigcap hX^+ = \varnothing$, $gX^+ \bigcap hX^- = \varnothing$, $gX^- \bigcap hX^- = \varnothing$, and $gX^- \bigcap hX^+ = \varnothing$. Therefore, $(\mathbf{g}\mathbf{X}^+ \bigcup (gX^- \setminus \{1\}) \bigcap (hX^+ \bigcup (hX^- \setminus \{1\}) =$ $(gX^+ \bigcap (hX^+ \bigcup (hX^- \setminus \{1\}) \bigcup$ $(gX^- \setminus \{1\}) \bigcap (hX^+ \bigcup (hX^- \setminus \{1\})) = \varnothing$. Then $gX^+ \bigcup (gX^- \setminus \{1\}) \subseteq B$. Similarly, we have $hX^+ \bigcup (hX^- \setminus \{1\}) \subseteq A$. Since $X^* = gX^+ \bigcup gX^- \bigcup A = hX^+ \bigcup hX^- \bigcup B$, then $X^+ = gX^+ \bigcup ($ $gX^- \setminus \{1\}) \bigcup A \bigcup hX^+ \bigcup (hX^- \setminus \{1\}) \bigcup B \subseteq A \bigcup B$. Then $X^+ \subseteq A \bigcup B$. Hence $X^+ = A \bigcup B$. □

Theorem 2.8. Let $A, B \subseteq X^+$ be two maximal left singular languages and $g \in l(A), h \in l(B)$. If $g \neq h$, then $A \bigcup B$ is not a left singular language.

Proof. If the first letters of g, h are different, then by proposition 2.7, we know $A \bigcup B = X^+$ which is not a left singular language. If the first letters of g, h are the same, let $g = ax_1, h = ax_2$ for some $a \in X, x_1, x_2 \in X^*$ and $x_1 \neq x_2$. Suppose $A \bigcup B$ is a left singular language. There exists a left singular word $f \in A \bigcup B$, that is to say, $f \in l(A \bigcup B)$. Therefore, $f \in A$ or $f \in B$. Without loss generality, we may assume $f \in A$. Then $f \in l(A)$. Since A is a maximal left singular language, then A only have one left singular word by lemma 1.1. So $g = f$. If $g \leq_p h$ or , then $h \leq_p g$ or $hx_3 = g$ $gx_3 = h$ for some $x_3 \in X^*$, which contradicts. If $g \not\leq_p h$ and $h \not\leq_p g$, then $gX^+ \bigcap hX^+ = \varnothing$ and $gX^+ \bigcap hX^- = \varnothing$. Since $X^* = hX^+ \bigcup hX^- \bigcup B$, we have $gX^+ \subseteq B$. Hence $\{g, gX^+\}$, $\subseteq A \bigcup B$ which contradicts $g \in l(A \bigcup B)$. If $f \in A$ and $f \in B$, since every maximal left singular language only have a left singular word, then $f = g = h$, which contradicts $g \neq h$. Thus $A \bigcup B$ is not a left singular language. □

For example: let $X = \{a, b\}$ and let $A = \{a^3,\ a^2bX^*,\ ab^2X^*,\ bX^*\}$, $B = \{a, bX^*\}$ be two maximal left singular languages. Then we can see $A \bigcup B = \{bX^*, a^2bX^*, a,\ ab^2X^*,\ a^3\}$ has no left singular word. In fact, by the former two results, we know that the union of two maximal left singular languages is a left singular language if and only if the left singular words of these maximal left singular languages are the same word.

Theorem 2.9. If $A, B \subseteq X^+$ are two maximal left singular languages and $g \in l(A)$, $h \in l(B)$, then $A \bigcap B \neq \varnothing$.

Proof. Since $A, B \subseteq X^+$ are two maximal left singular languages and $g = f \in l(A \bigcup B)$ $g \in l(A), h \in l(B)$, then $X^* = gX^+ \bigcup gX^- \bigcup A = hX^+ \bigcup hX^- \bigcup B$ by lemma 2.5. We consider the following two cases.

If the first letters of g and h are the same, let $g = ax_1$, $h = ax_2$ for some $a \in X$ and $x_1, x_2 \in X^*$. We consider the word b where $b \in X$ and $b \neq a$. Since $b \notin gX^+$, $b \notin gX^-$ and $b \notin hX^+$, $b \notin hX^-$, then $b \in A$ and $b \in B$. So $A \bigcap B \neq \varnothing$.

If the first letters of g and h are different, let $g = ax_3, h = ax_4$ for some $a, b \in X$, $b \neq a$ and $x_3, x_4 \in X^*$. Since the first letter of g is a, then $bX^* \bigcap (gX^+ \bigcup gX^-) = \varnothing$. Since $X^* = gX^+ \bigcup gX^- \bigcup A$, we have $bX^* \subseteq A$. Hence $bx_4 \in A \bigcap B$. Thus $A \bigcap B \neq \varnothing$. □

Let $w \in X^+$ and $w = a_1 a_2 \cdots a_{n-1} a_n$ where $a_i \in X$ for $i = 1, 2, \cdots, n$. We denote $\tilde{w} = a_n a_{n-1} \cdots a_2 a_1$. Let $A = \{\tilde{w} | \text{for any } w \in A\}$ (see [2]), which is called the image of A, If a prefix code P, then $\tilde{P}$ sometimes is a prefix code, sometimes is not a prefix code. Then how about maximal singular language, then $\tilde{A}$ is not a maximal left singular language.

Proposition 2.10. If A is a maximal left singular language, then $\tilde{A}$ is not a left singular language.

Proof. Since A is a maximal left singular languages and $g \in l(A)$, then $X^* = gX^+ \bigcup gX^- \bigcup A$, Let $g = ax \in aX^*$ where $a \in X$. Then $bX^* \subseteq A$ for any $b \in X$ and $b \neq a$. So $bg \in A$. We know $\tilde{bg} = \tilde{g}b \in \tilde{A}$. Since $\tilde{g} \in \tilde{A}$, then $\tilde{g}$ is not a left singular word in $\tilde{A}$. For any $\tilde{z} \in \tilde{A} \backslash \{\tilde{g}\}$, we have $z \in A \backslash \{g\}$. Since $bz \in A$ and $\tilde{zb} = \tilde{z}b \in \tilde{A}$, then $\tilde{z}$ is not a left singular word in $\tilde{A}$. Thus $\tilde{A}$ is not a left singular language. □

Let $X = \{a, b\}$ and $A = \{a, bX^*\}$ be a maximal left singular language. Then $a \in l(A)$, $aX^- = \{1\}$. So $X^* = aX^+ \bigcup \{1\} \bigcup A$. Then $\tilde{A} = \{a, X^* b\}$. Since $a, ab \in \tilde{A}$, then $a \notin l(A)$. We can see $\tilde{A}$ is not a left singular language.

A maximal left singular language must be a left singular language, but not every left singular is maximal. As we know, every prefix code is contained in a maximal prefix code. At last of the paper, we want to know whether every left singular language is contained in a maximal left singular language or not?

Proposition 2.11. Let A be a left singular language. Then there exists a maximal left singular language M such that $A \subseteq M$.

Proof. If A is maximal left singular language, then $M = A$. If A is a left singular language, but it is not a maximal left singular language. Let $g \in l(A)$. Let $M = X^* \backslash (gX^+ \bigcup gX^-)$. Since $g \notin gX^+ \bigcup gX^-$, then $g \in l(M)$ and $X^* = gX^+ \bigcup gX^- \bigcup M$. So we know M is a maximal left singular language. For any $x \in A$, we consider the following cases. If $x = g$, then $x \in M$. If $x \neq g$, since $g \in l(A)$, then $x \notin gX^+ \bigcup gX^-$. So $x \in M$. Thus $A \subseteq M$. □

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