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Data Dispersion in Economics(II)---
Inevitability and Consequences of Restrictions

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Abstract: This article reviews and improves the theorems of the existence of restrictions near the boundaries of finite numerical segments and of the probability scale in the presence of non-zero dispersion. The non-zero dispersion may be caused, for example, by the influence of observation noises. Applications of the theorems to experiments, which are typical of the utility theory, are briefly presented. Similar experiments may be associated with the old problems of utility theory, such as the underweighting of high and the overweighting of low probabilities, risk aversion, loss aversion, the Allais paradox, the equity premium puzzle, the "four-fold pattern" paradox, etc. It is shown that the restrictions as the consequences of the theorems should be taken into account in the explanation of such experiments. The restrictions may facilitate such explanations including explanations by utility models.

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1. Introduction

The review concerns basic utility theory problems such as the Allais paradox. In 2006 these problems remained unsolved (see Kahneman and Thaler (2006)). The main idea of the review is as follows.

Regarding data dispersion, restrictions can exist for the mean values near the boundaries of finite numerical segments and near the boundaries of the probability scale.

The restrictions near the boundaries of the probability scale can bias the results of experiments and can help to explain them partially.

The review is based on the proofs of the theorems (Harin (2010a, 2011)) and their applications (Harin (2010b, 2012a)). The theorems support the earlier works by Harin (2003 (in Russian), 2005 (in English) and 2009b). The regular monitoring of the scholarly literature by Harin (2005–2011, 2007–2011a and 2007–2011b) confirms that the theorems are original.

2. Utility Theory and Its Problems

2.1 Prehistory

Human decisions are fundamental constituents of the economy and economic theory and specifically of utility theory. The problems of utility theory started with Bernoulli (1738). A sharp worsening of the situation occurred when Von Neumann and Morgenstern (1947) provided promises of the feasibility of correct and, naturally, rational fundamentals of economic theory, but Allais (1953) broke these promises. Numerous later works supported the statement that people's decisions are inconsistent with rational suppositions. Kahneman and Thaler (2006) summarized these inconsistencies as having still not been overcome by economic theory.

Many works (see, e.g., Ellsberg (1961), Kahneman and Tversky (1979), Schoemaker (1982), Tversky and Wakker (1995), Prelec (1998), Gilboa (2010), Robson and Samuelson (2011)) have been undertaken in this field. A sketchy overview of some basic problems is presented below.

2.2 Examples of Trouble in Utility Theory

Let us consider three examples of the basic problems of utility theory (the same ones as in the first part of this review). Before the examples, we will pay attention to the concept of the prediction of mean value and we will define the underestimation and overestimation of probable outcomes.

Let the underestimation of probable outcomes be the negative and the overestimation the positive deviation from the mean value prediction.

2.2.1 Gain at high probabilities

Let us presume that a scholar offers a choice of two outcomes:

A) a guaranteed gain of a prize of *$99* (with the probability *1* or *100%*)

or

B) a probable gain of *$100* with the probability *0.99* (or *99%*), or nothing with the probability *0.01* (or *1%*).

For experimental accuracy, both *$99* and *$100* should be in *$1* banknotes, i.e. *99* and *100* banknotes of *$1*. The mean values for the probable and guaranteed outcomes are

$$\$99 \times 100\% = \$99,$$
$$\$100 \times 99\% = \$99.$$
$$\text{Here, } \$99 = \$99.$$

The mean value for obtaining the probable gain is evidently precisely equal to the mean value for obtaining the guaranteed gain. However, the overwhelming majority of people chose the guaranteed gain instead of the probable one in such experiments (see, e.g., Tversky and Wakker (1995)).

So, people underestimate the probable outcomes and do not like risk. The deviation from the prediction of the mean value is negative. Risk aversion is a well-known explanation for this problem. However, this explanation fails when the gain is changed to loss or high probability is changed to low probability. Let us consider these cases.

2.2.2 Losses at high probabilities

Let us presume that a scholar offers a choice of two outcomes:

A) a guaranteed loss of *-$99* (with the probability *100%*)

or

B) a probable loss of *-$100* with the probability *99%*, or no loss with the probability *1%*.

The mean values for the probable and guaranteed outcomes are

$$-\$99 \times 100\% = -\$99,$$
$$-\$100 \times 99\% = -\$99.$$
$$\text{Here, } -\$99 = -\$99.$$

The mean value for getting the probable loss is evidently precisely equal to the mean value for getting the guaranteed loss. However, the overwhelming majority of people chose the probable loss instead of the guaranteed one in such experiments (see, e.g., Tversky and Wakker (1995)).

Thus, people overestimate the probable outcomes and like risk. The deviation from the prediction of the mean value is positive.

2.2.3 Gains at low probabilities

Let us presume that a scholar offers a choice of two outcomes:

A) a guaranteed gain of *$1* (with the probability *100%*)

or

B) a probable gain of *$100* with the probability *1%*, or nothing with the probability *99%*.

The mean values for the probable and guaranteed outcomes are

$$\$1 \times 100\% = \$1,$$
$$\$100 \times 1\% = \$1.$$
$$\text{Here, } \$1 = \$1.$$

The mean value for obtaining the probable gain is evidently precisely equal to the mean value for obtaining the guaranteed gain. However, most people chose the probable loss instead of the guaranteed one in such experiments (see, e.g., Tversky and Wakker (1995)).

Therefore, people overestimate the probable outcomes and like risk. The deviation from the prediction of the mean value is positive.

2.3 Conflict of Choices: Various Points of View

We see that the prevalent choices of people conflict with each other and, moreover, are contrary to each other.

Indeed, people do not like risk relating to gains at high probabilities. However, they like risk regarding gains at low probabilities and losses at high probabilities. These choices cause a large amount of unexplained problems and paradoxes.

We see that the real personal attitude to risk is not captured by the concept of the prediction of mean value. There are two main points of view concerning this conflict. The first supposes that it is stimulated by the lack of describing power of the expected utility theory. The second (see, e.g., Kahneman and Tversky (1979)) supposes that it is stimulated by the reason that people are not fully rational.

However, some authoritative scholars hold a different point of view (see, e.g., Hey and Orme (1994), Chay, McEwan and Urquiola (2005), Hey (2005), Butler and Loomes (2007)). This point of view pays more attention to noise and imprecision.

Harin (2012b), as the other part of this review, supports this point of view and pays attention to the dispersion of data.

2.4 From Possibility to Inevitability of the Existence of Restrictions

Two statements have been proved by Harin (2012b):

1) Regarding the condition of non-zero dispersion of data, non-zero restrictions can exist near the boundaries of numerical segments.

2) Regarding the condition of non-zero dispersion of data, non-zero restrictions can exist near the boundaries of the probability scale.

The following theorems of existence should specify the conditions, namely the conditions of the dispersion of data, in which the restrictions must exist.

3. Theorem of the Existence of Restrictions near the Boundaries of Numerical Segments

3.1 General Preliminary Notes

3.1.1 General conditions, assumptions and notations

Suppose a numerical segment $X=[A; B]$: $0<(B-A)<\infty$ and a quantity $\{x_k\}$: $A\le x_k\le B$, $k=1, 2, \ldots K$: $2\le K\le\infty$, with the weights $w_K(x_k)$: $w_K(x_k)\ge 0$, and

$$\sum_{k=1}^{K} w_K(x_k) = W_K ,$$

where W_K (the total weight of $\{x_k\}$) is such that $0 < W_K < \infty$.

Keeping generality, $w_K(x_k)$ may be further normalized so that $W_K=1$. A more general case actually plays no role in the article.

For this article, let us denote a moment of n-th order $E[(X-X_0)^n]$ of the quantity $\{x_k\}$ as

$$E[(X - X_0)^n] = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - x_0)^n w_K(x_k)$$

Suppose the initial moment of the first order, the mean value $E[X]$ of the quantity $\{x_k\}$, exists:

$$E[X] = \frac{1}{W_K}\sum_{k=1}^{K} x_k w_K(x_k) \equiv M$$

Suppose for n : $1<n<\infty$ that at least one central moment of the quantity $\{x_k\}$ exists:

$$E[(X - M)^n] = \frac{1}{W_K}\sum_{k=1}^{K}(x_k - M)^n w_K(x_k) .$$

3.1.2 Maximal possible value of a central moment

The maximal possible value of a central moment may be estimated from its definition:

$$|E[(X-M)^n]| \equiv \left|\frac{1}{W_K}\sum_{k=1}^{K}(x_k - M)^n w_K(x_k)\right| \le \frac{1}{W_K}\sum_{k=1}^{K}|(x_k - M)^n| w_K(x_k) \le$$

$$\le \frac{1}{W_K}(B-A)^n \sum_{k=1}^{K} w_K(x_k) = (B-A)^n$$

A more precise estimation of this value is provided (see, e.g., Harin (2010a or 2011)) by the sum of the modules of the central moments of the functions that are concentrated at the boundaries of the segment:

$$Max(|E[(X-M)^n|]) \le \left|(A-M)^n \frac{B-M}{B-A}\right| + \left|(B-M)^n \frac{M-A}{B-A}\right| .$$

3.2 General Lemma about the Tendency to Zero for Central Moments

Lemma: If, for $\{x_k\}$, defined in section 3.1.1, $M\equiv E[X]$ tends to A or to B, then, for $1<n<\infty$, $|E[(X-M)^n]|$ tends to zero.

Proof: For $M\to A$,

$$|E[(X-M)^n]| \le |(A-M)^n \frac{B-M}{B-A}| + |(B-M)^n \frac{M-A}{B-A}| \le$$
$$\le [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \le$$
$$\le 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0$$

So, if *(B-A)* and *n* are finite and $M \to A$ (that is, $(M-A) \to 0$), then $|E[(X-M)^n]| \to 0$.

For $M \to B$, the proof is similar.

This lemma has been proved.

Note. More precisely (see, e.g., Harin (2011)), the estimation may be obtained for central moments' tendency to zero, e.g. for $M \to A$.

$$|E[(X-M)^n]| \le (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 .$$

3.3 General Theorem of the Existence of Restrictions for Mean Values

Theorem: If there are: a quantity $\{x_k\}$ defined as in section 3.1, $n : 1<n<\infty$ and $r_{dispers}>0 : |E[(X-M)^n]| \ge r_{dispers}$, then a restriction $r_{mean}>0$ exists: $A<(A+r_{mean}) \le E[X] \le (B-r_{mean})<B$.

Proof: From the lemma, for $M \to A$,

$$0 < r_{dispers} \le |E[(X-M)^n]| \le 2(B-A)^{n-1}(M-A),$$

$$0 < \frac{r_{dispers}}{2(B-A)^{n-1}} \le (M-A)$$

and

$$r_{mean} \equiv \frac{r_{dispers}}{2(B-A)^{n-1}} .$$

For $M \to B$, the proof is similar.

As long as *(B-A)*, *n* and $r_{dispers}$ are finite and $r_{dispers}>0$, then r_{mean} is finite, $r_{mean}>0$, both $(M-A) \ge r_{mean}>0$ and $(B-M) \ge r_{mean}>0$. This theorem has been proved.

So, if a finite $(n<\infty)$ central moment of a quantity, which is defined for a finite segment, cannot approach closer to *0* than by a non-zero value $r_{dispers}>0$, then the mean value of the quantity also cannot approach closer to a boundary of this segment than by the non-zero value $r_{mean}>0$.

More generally, if a quantity is defined for a finite segment and a non-zero restriction $r_{dispers}>0$ exists between zero and the zone of possible values of a finite ($n<\infty$) central moment of the quantity, then the non-zero restrictions $r_{mean}>0$ also exist between a boundary of the segment and the zone of possible values of the mean of this quantity.

4. Theorems of the Existence of Restrictions near the Boundaries of the Probability Scale

4.1 General Notes

Let there be a probability estimation, frequency F_K of an event, and $\{x_k\}$ be the results of a series of tests that give us this probability estimation, frequency F_K of the event, so that $\{x_k\}$ is such that $E[X] \equiv M \equiv F_K$ for $\{x_k\}$. Let $\{x_k\}$ has the characteristics defined in section 3.1.1; in particular, $\{x_k\}$ is defined: 1) for a series of tests of number $K : K>>1$, 2) for the probability scale $[A; B]=[0; 1]$ (in the probability-theoretic notation, or, in the common notation $[0\%; 100\%]$) and 3) for $W_K=1$.

4.2 Lemma about the Tendency to Zero for Probability Estimation

Lemma: If $\{x_k\}$ is defined as in section 4.1, and either $E[X]\rightarrow 0$ or $E[X]\rightarrow 1$, then, for $1<n<\infty$, $|E[(X-M)^n]|\rightarrow 0$.

Proof: As long as the conditions of this lemma satisfy the conditions of the lemma in section 3.2, then the statement of this lemma is as true as the statement of the lemma in section 3.2.

This lemma has been proved.

4.3 Theorem of the Existence of Restrictions for Probability Estimation

Theorem: If a probability estimation, frequency F_K, and $\{x_k\}$ are defined as in section 4.1, such that $M\equiv E[X]\equiv F_K$, there are $n : 1<n<\infty$, and $r_{dispers}>0 : E[(X-M)^n]\geq r_{dispers}>0$, then, for the probability estimation, frequency $F_K\equiv M\equiv E[X]$, a restriction r_{mean} exists such as $0<r_{mean}\leq F_K\leq(1-r_{mean})<1$.

Proof: As long as the conditions of this theorem satisfy the conditions of the theorem in section 3.3, then the statement of this theorem is as true as the statement of the theorem in section 3.3.

This theorem has been proved.

4.4 Theorem of the Existence of Restrictions for Probability

Theorem: If, for the probability scale $[0; 1]$, a probability P and the probability estimation, frequency F_K, for a series of tests of number $K : K>>1$, are determined such that when the number K of tests tends to infinity, the frequency F_K tends at that to the probability P, that is

$$P = \lim_{K\rightarrow\infty} F_K$$

non-zero restrictions $r_{mean} : 0<r_{mean}\leq F_K\leq(1-r_{mean})<1$ exist between the zone of the possible values of the frequency and every boundary of the probability scale, then the same non-zero restrictions $r_{mean} : 0<r_{mean}\leq P\leq(1-r_{mean})<1$ exist between the zone of the possible values of the probability P and every boundary of the probability scale.

Proof: Consider the left boundary 0 of the probability scale $[0; 1]$. The frequency F_K is not less than r_{mean}:

$$F_K \geq r_{mean}$$

Hence, we obtain for P:

$$P = \lim_{K\rightarrow\infty} F_K \geq \lim_{K\rightarrow\infty} r_{mean} = r_{mean}\,.$$

So, $P\geq r_{mean}$. Note that this is true for both monotonous and dominated convergence. The reason is the fixation of the minimal value of all the F_K by the conditions of the theorem.

For the right boundary 1 of the probability scale the proof is similar to that above.

This theorem has been proved.

5. Consequences of Restrictions

5.1 Dispersion of Data and Restrictions

5.1.1 Background noises, measurement errors, etc.

Real measurements of probability are almost always performed in the environment of non-zero external interference, background noises, disturbances, etc. This leads to the finite non-zero external uncertainty of measurements.

5.1.2 The total uncertainty

Thus, in almost any real case, a finite non-zero degree of uncertainty is inherent in real measurements of probability. The total magnitude of this uncertainty can be both negligible and

high, relative to a useful signal, but it is finite and non-zero (it does not tend to zero).

5.1.3 Dispersion of data and restrictions

This uncertainty leads to the non-zero dispersion of data, namely of the probability estimation values, for all such cases. Thus, in almost any real case non-zero dispersion of the probability estimation values exists. This dispersion leads to the non-zero restrictions in the probability scale. The calculation (Harin (2009a)) gives the value of the restriction as not less than *1/3* of the root-mean-square value of the dispersion for the normal distribution of the dispersion.

Therefore, in almost any real case, the non-zero restrictions exist for the possible values of the probability.

5.2 Consequence of the Existence of Restrictions

5.2.1 Probability cannot be located in a restriction

The essence of a restriction is the statement "a mean value cannot be located in a restriction, in a restricted zone." A probability estimation, frequency (as a mean value), cannot be located in a restriction either. The probability, as the limit of the probability estimation, also cannot be located in the restriction.

Non-zero dispersion of data causes non-zero restrictions near any boundary of the probability scale. Let us define a non-zero restriction as $r_{Restriction}$, such as $r_{Restriction}>0$.

Suppose the dispersion of data leads to restrictions that are not less than, say, $r_{Restriction}\geq 3\%$ (in the common notation or $r_{Restriction}\geq 0.03$ in the probability-theoretic notation). Then, near the left boundary, *0%* of the probability scale [*0%; 100%*], the probability cannot be equal to, say, *1%* and is not less than *3%*. Near the right boundary, *100%* of the probability scale [*0%; 100%*], the probability cannot be equal to, say, *99%* and is not more than *97%*.

5.2.2 Ideal and real cases

Suppose two cases:

1) An ideal case. There is no (or neglible) dispersion of data. Hence, the probability may be equal to any value near any boundary of the probability scale. Suppose a value $P_{Ideal}=100\%-\delta$: $0<\delta<<100\%$, of the probability located near *100%* in an ideal case with zero dispersion of data.

The scholar performing the experiment may keep in mind the ideal case of no (or neglible) dispersion of data and may propose probabilities that are very close to the boundary of the probability scale.

2) A real case. There is non-zero dispersion of data. The non-zero dispersion of the data causes a non-zero restriction, say $r_{Restriction}\geq 3\%$, near any boundary of the probability scale.

If the whole real experience of people proves that the dispersion is usually large, then the people, contrary to the scholar, may keep in mind the real case of the large dispersion, namely of the dispersion that causes a non-zero restriction, say $r_{Restriction}\geq 3\%$.

5.2.3 Bias of probability, overcoming the influence of observation noise, and Biases for high and low probabilities

Suppose a transformation from an ideal to a real case. This corresponds to the transformation from the point of view of the scholar to the point of view of the people.

The absence or neglible dispersion of data is transformed to non-zero dispersion of data. A value of the probability in the ideal case P_{Ideal} will be transformed to that of the real case P_{Real}.

Let a restriction in the probability scale be increased from the ideal case of zero to the real case of some non-zero magnitude, say to $r_{Restriction}=3\%$. The probability P_{Ideal} is transformed to some probability P_{Real}.

Consider the probability near the right boundary, *100%* of the probability scale [*0%; 100%*]. The probability cannot be located in the restriction. Hence, the probability P_{Real} cannot be more than $P_{Real} \leq 100\%-r_{Restriction}=100\%-3\%=97\%$. If the dispersion of data is increased to the extent that the restriction exceeds the difference between *100%* and P_{Ideal}, that is, $r_{Restriction}>\delta$, then P_{Real} cannot be equal to (or more than) P_{Ideal}. The ideal case probability of, say, *98%* cannot be located in the restriction and is biased to a position that is not more than *97%*. Every particular ideal probability from *97.000...01%* to *99.999...99%* is also biased to a corresponding real position that is not more than *97%*.

So, near *100%*, P_{Real} is biased downward to the middle of the probability scale with respect to P_{Ideal}, that is, near *100%*, $P_{Real}<P_{Ideal}$. The closer the probability P_{Ideal} is to *100%*, the greater is the bias $P_{Ideal}-P_{Real}$. Conversely, for any non-zero restriction $r_{Restriction}>0$, a $P_{Ideal}=100\%-\delta$: $\delta>0$ will exist, such as $r_{Restriction}>\delta>0$, and, hence, $P_{Real}<P_{Ideal}$.

An analogous consideration may be performed for a probability located near *0%*.

So, the restrictions near the boundaries shift and bias the probability from the boundaries to the middle of the probability scale. The bias is directed to the middle and is maximal just near every boundary.

Therefore, in the ideal case (or from the point of view of the scholar), the probability is unbiased. In the real case (or from the point of view of the people), the probability near every boundary is biased (in comparison with the ideal case) from the boundary to the middle of the probability scale. Taking into account the restrictions and the biases may help to overcome the influence of observation noise, to refine the results of experiments from this influence.

Note that the bias may be supposed to exist not only in the zones of the restrictions but also beyond them and to vanish at the middle of the scale.

Note also that the signs of the biases of the probability are opposite for high and low probabilities. The sign of the bias is negative for high probabilities and positive for low probabilities. So, according to the mean value theorem, in some point at the middle of the scale the bias should be equal to zero.

5.2.4 Biases for gain and loss

The bias of probability causes the bias of the mean result of an outcome. Indeed, the mean result is the product of the value of gain or loss and of the probability of this result. The bias of a co-factor causes the bias of the product, while another co-factor remains constant.

The signs of the bias of mean results are opposite for gains and for losses. The reason is that the sign (positive) of gains' values is opposite with respect to the sign (negative) of losses' values. Changing the sign of a co-factor causes the sign of the product to change, while the sign of another co-factor remains constant.

5.2.5 Two reasons for changing the sign: biases and experiments

We obtain two reasons for changing the sign of the mean results of outcomes:

1) The sign of outcomes is changed when we change high probabilities to low probabilities and vice versa.
2) The sign of outcomes is changed when we change gains to losses and vice versa.

At high probabilities, the sign of the bias is negative for gains and positive for losses. At low probabilities, the sign of the bias is positive for gains and negative for losses.

The signs of these biases of mean results successfully coincide with the signs of the biases of real experiments. So, when the values of these biases are less than the values of the biases of experiments, we may expect these biases partially to explain the real experiments. So, a uniform

point of view and a uniform approach may be maintained in all these experiments.

Rigorously speaking, these biases explain the experiments partially or at least partially. However, explaining them completely was not a goal of the research of this article. The goal was to reveal an unknown source of the paradoxes and problems: it was not to provide a complete solution to them, but to help to search for such a solution.

These paradoxes and problems include, for example, the underweighting of high probabilities and the overweighting of low probabilities, risk aversion, the equity premium, the Allais paradox, the "four-fold pattern" paradox, etc. (see, e.g., Harin (2007)). The idea of the bias was presented for the first time as a hypothesis, as an assumption by Harin (2003) in Russian and Harin (2005) in English. This assumption was proved by Harin (2010a).

6. Partial Explanation of the Problems

Let us apply the above considerations to the examples of problems described in section 2.

6.1 Preliminary Notes

First, we should specify the conditions "probable gain" and "probable loss" of the examples. Let us name them generally "probable outcomes." Due to the above considerations, we should distinguish between the two cases:

1) Probable outcomes of the ideal case (from the point of view of the scholar), and

2) Probable outcomes of the real case (from the point of view of the people).

Concerning these "probable outcomes":

1) The scholar may keep in mind the ideal case of neglible dispersion of data and may propose probabilities that are very close to the boundary of the probability scale.
2) People may keep in mind the real case of large dispersion, namely the dispersion that causes a non-zero restriction, say $r_{Restriction} \geq 3\%$.

If the whole real experience of the people proves to them that in real life $P<97\%$, then the people will keep in mind the real $P<97\%$, though the scholar will keep in mind $P=99\%$. If the whole real experience of the people proves to them that in real life $P>3\%$, then the people will keep in mind the real $P>3\%$, though the scholar will keep in mind $P=1\%$.

Moreover, if the scholar says $P=99\%$, the people nevertheless keep in mind the real $P<97\%$, e.g. supposing that $P=99\%$ is the ideal probability without the real large dispersion. If the scholar says $P=1\%$, the people nevertheless keep in mind the real $P>3\%$, e.g. supposing that $P=1\%$ is the ideal probability without the real large dispersion.

So, let us suppose that people keep in mind $3\%<P<97\%$ irrespective of what the scholar keeps in mind or says and pass to the examples of problems in section 2.2.

6.2 Gain at High Probabilities

Let us presume that a scholar offers a choice of two outcomes:

A) a guaranteed gain of *$99* (with the probability *100%*)

or

B) a probable gain of *$100* with the probability *99%*.

In the ideal case and from the point of view of the scholar, the probable gain has the probability *99%* and the mean values for the probable and guaranteed outcomes are

$$\$99 \times 100\% = \$99,$$
$$\$100 \times 99\% = \$99.$$
$$\text{Here, } \$99 = \$99.$$

So, in the ideal case and from the point of view of the scholar, the mean of the probable gain and the mean of the guaranteed gain are both equal to *$99* and are precisely equal to each other; the two outcomes are equally preferable.

In the real case and from the point of view of the people, if the dispersion of real data leads to the restriction (near *100%*) that is more than *1%* and is equal to, say, *3%*, then the probability of the probable gain cannot be equal to *99%* and is not more than *97%*

$$\$99 \times 100\% = \$99,$$
$$\$100 \times 97\% = \$97.$$
$$\text{Here, } \$99 > \$97.$$

So, the mean value of the probable gain is less than the mean value of the guaranteed gain.

So, in the real case and from the point of view of the people, the mean of the probable gain is less than the mean of the guaranteed gain and the guaranteed outcome is preferable.

So, the paradox can be explained, at least partially, by taking into account the restriction and the bias of the mean result.

Although we have not considered any particular utility model, this result may be useful for decision and utility theories on the whole as a correction of used probability.

6.3 More Than One Field: Uniform Point of View

Let us reveal and summarize the results of the application of the theorem in two more fields.

6.3.1 Loss at high probabilities

Let us presume that a scholar offers a choice of two outcomes:

A) a guaranteed loss of *-$99* (with the probability *100%*)

or

B) a probable loss of *-$100* with the probability *99%*.

In the ideal case and from the point of view of the scholar, the probable loss has the probability *99%* and the mean values for the probable and guaranteed outcomes are

$$-\$99 \times 100\% = -\$99,$$
$$-\$100 \times 99\% = -\$99.$$
$$\text{Here, } -\$99 = -\$99.$$

So, in the ideal case and from the point of view of the scholar, the mean of the probable loss and the mean of the guaranteed loss are both equal to *-$99* and are precisely equal to each other; the two outcomes are equally preferable.

In the real case and from the point of view of the people, if the dispersion of real data leads to the restriction (near *100%*) that is more than *1%* and is equal to, say, *3%*, then the probability of the probable loss cannot be equal to *99%* and is not more than *97%*.

$$-\$99 \times 100\% = -\$99,$$
$$-\$100 \times 97\% = -\$97.$$
$$\text{Here, } -\$99 < -\$97.$$

So, the mean value of the probable loss is more than the mean value of the guaranteed loss (it is less in the absolute value but more due to the negative sign of the loss).

So, in the real case and from the point of view of the people, the mean of the probable loss is more than the mean of the guaranteed loss and the probable outcome is preferable.

So, the paradox can be explained, at least partially, by taking into account the restriction and the bias of the mean result.

6.3.2 Gain at low probabilities

Let us presume that a scholar offers a choice of two outcomes:

A) a guaranteed gain of *$1* (with the probability *100%*)

or

B) a probable gain of *$100* with the probability *1%*.

In the ideal case and from the point of view of the scholar, the probable gain has the probability *1%* and the mean values for the probable and guaranteed outcomes are

$$\$1 \times 100\% = \$1,$$
$$\$100 \times 1\% = \$1.$$
$$\text{Here, } \$1 = \$1.$$

So, in the ideal case and from the point of view of the scholar, the mean of the probable gain and the mean of the guaranteed gain are both equal to *$1* and are precisely equal to each other; the two outcomes are equally preferable.

In the real case and from the point of view of the people, if the dispersion of real data leads to the restriction (near *100%*) that is more than *1%* and is equal to, say, *3%*, then the probability of the probable gain cannot be equal to *1%* and is not less than *3%*.

$$\$1 \times 100\% = \$1,$$
$$\$100 \times 3\% = \$3,$$
$$\text{Here, } \$3 > \$1.$$

So, the mean value of the probable gain is more than the mean value of the guaranteed gain.

In the real case and from the point of view of the people, the mean of the probable gain is more than the mean of the guaranteed gain and the probable outcome is preferable.

Thus, the paradox can be explained, at least partially, by taking into account the restriction and the bias of the mean result.

6.3.3 Hypothesis of small dispersion and biases

Let us suppose a hypothesis:

The dispersion of data is small as all the biases, which are caused by the restrictions due to the dispersion, are not more than the corresponding experimental biases.

For this hypothesis, all the biases, which are caused by the restrictions, due to their small absolute values and to their coincidence of the signs with the experimental biases, reduce all the experimental biases that should be explained. So, this may reduce the absolute values of the examples' paradoxes that should be explained.

Hence, this may facilitate the explanation of the considered examples' paradoxes, including explanations by utility models.

Note that this reduction facilitation is performed from the unified point of view for essentially different examples.

(Note also, the not great asymmetry between gains and losses should be taken into account. This asymmetry is observed in practice and has been explained (see Harin (2007)) in the scope of the proposed approach.)

6.3.4 More than one field: uniform point of view

We see that the results of the application of the theorem may be applied not only in one field of gains at high probabilities. Taking into account the restrictions and the biases of the mean result, the

results of non-zero dispersion of data may be applied in more than one field.

Taking into account the restrictions and the biases may help to overcome the influence of observation noise and to refine the results of the experiments from this influence from the uniform point of view.

Regarding the hypothesis of small dispersion and biases, taking into account the restrictions and the biases may partially explain the results of experiments in more than one field from the uniform point of view.

The above considerations do not deal with any general or particular utility or prospect model. Nevertheless, these considerations show that the probability should be corrected near the boundaries of the probability scale in the real prevalent presence of data dispersion. This correction should be taken into account in every utility and prospect model and may be useful for them.

7. Conclusion

The inequalities of distribution moments and theorems of existence of restrictions near the boundaries of finite numerical segments and of the probability scale are presented.

The consequences of the inequalities and theorems may have applications in economics and finance, including:

1) The restrictions should exist for the probability near the boundaries of the probability scale in the presence of non-zero dispersion of data, which may be caused, for example, by the influence of observation noise. These restrictions lead to the biases, which should be taken into account when considering the results of experiments near the boundaries of the probability scale, including explanations of these results by utility models.

Taking into account the restrictions and the biases may help to overcome the influence of observation noise and to refine the results of the experiments from this influence.

2) Regarding the hypothesis of small dispersion and biases, taking into account the restrictions and the biases diminishes the absolute values of the paradoxes and, therefore, may help partially to explain these paradoxes.

3) Concerning the hypothesis of small dispersion and biases, taking into account the restrictions and the biases, as the results of non-zero dispersion of data, may partially explain the results of the experiments in more than one field from the uniform point of view.

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