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A Consistent Allocation Rule: Non-emptiness, Reductions, Domination and Axiomatization

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Abstract: Since an extended core violates related properties of consistency, this paper proposes the concept of pseudo core. By focusing on the agents and the operational levels simultaneously, an extended reduction is also introduced to axiomatize the pseudo core. Furthermore, this paper adopts the duality results of linear programming theory to determine the non-emptiness of the pseudo core. By investigating the domination among payoff vectors, this article also proposes the dominance core and related coincidences between the pseudo core and the dominance core.

JEL Classifications: C70, C71, D70

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1. Introduction

In a sum and in short, the *core* has been wildly applied among the allocation concepts on traditional games. Besides, related properties of *consistency* and *converse consistency* play key roles in various axiomatizations of the core. Based on the notion of the *reduced game*, these two properties have been investigated comprehensively. *Consistency* allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restriction to each subgroup for the associated reduction the subgroup faces; as its name indicates, *converse consistency* permits a converse operation: to deduce the desirability of an outcome for some problem from the desirability of its restriction to each subgroup for the associated reduction this subgroup faces. The Davis and Maschler's (1965) *max-reduction* and the Moulin's (1985) *complement-reduction* are two primary reductions for the axiomatizations of the core on traditional games. Related axiomatic results could be found in Peleg (1985, 1986), Tadenuma (1992), Serrano and Volij (1998), Voorneveld and Nouweland (1998) and Hwang and Sudhölter (2001), and so on.

In the framework of traditional games, each agent is either completely involved or excluded at all in participation with some other agents. In the framework of *multi-choice transferable-utility (TU) games*, each agent is permitted to participate with finite various activity levels. In some conditions, there exists some classes of indivisible goods which are *only available in finite natural numbers*, such as cars, buildings, and so on. A multi-choice TU game is exactly a suitable game-theoretic method for modeling this condition. A multi-choice core was first proposed by Nouweland et al. (1995). By considering the agents only, Hwang et al. (2013) extended the max-reduction and the complement-reduction to multi-choice TU games, and showed that the core

due to Nouweland et al. (1995) violates related properties of consistency. Here we build on the results proposed by Nouweland et al. (1995) and Hwang et al. (2013). The main results of this paper are as follows.

1. Section 3 introduces the *pseudo core* by revising the core due to Nouweland et al. (1995). In Section 4, we adopt the duality results of linear programming theory to determine the *non-emptiness* of the pseudo core.

2. As we mentioned above, the agents and the operational levels are essential causes on multi-choice TU games. Different from the reductions due to Hwang et al. (2013), we propose a reduction by focusing on the agents and the operational levels simultaneously. In Section 4, we also adopt related properties of consistency and its converse to axiomatize the pseudo core. Further, we provide some more applications and comparisons in Sections 3 and 6.

3. In Section 5, we firstly adopt the notion of *domination* to propose the *dominance core*, and further show that the pseudo core coincides with the dominance core under some conditions.

2. Preliminaries

Let U be the universe of agents. For $i \in U$ and $b_i \in \mathbb{N}$, we set $B_i = \{0,1,\cdots,b_i\}$ to be the operational level collection of i, where 0 is dummy level. For $N \subseteq U$, let $B^N = \prod_{i\in N} B_i$ be the product set of the operational level collections of agents in N, and $K^N = \{(i,k_i) \mid i \in N, k_i \in B_i^+\}$, where $B_i^+ = B_i \setminus \{0\}$. In $\mathbb{R}^N$, denote the zero vector by 0_N.

A **multi-choice TU game** is a triple (N,b,v), where N with $0 < |N| < \infty$ is a set of agents, $b = (b_i)_{i\in N}$ is a vector that presents the amount of operational levels for each agent, and $v: B^N \to \mathbb{R}$ is a characteristic mapping which presents to each $\alpha = (\alpha_i)_{i\in N}$ the merit that the agents can gain when each agent i participates at operational level α_i with $v(0_N) = 0$. Denote the class of all multi-choice TU games by Γ.

Given $(N,b,v) \in \Gamma$, $\alpha \in B^N$ and $T \subseteq N$, we denote $\alpha_T \in \mathbb{R}^T$ to be the restriction of α to T. Let $i \in N$, we define α_{-i} to represent $\alpha_{N\setminus\{i\}}$ and let $\gamma = (\alpha_{-i}, k_i) \in \mathbb{R}^N$ be defined by $\gamma_{-i} = \alpha_{-i}$ and $\gamma_i = k_i$. Let $|T|$ be the amount of components in T, and let $\theta^T(N) \in \mathbb{R}^N$ be the vector satisfying $\theta_i^T(N) = 1$ if $i \in T$, and $\theta_i^T(N) = 0$ if $i \notin T$. $\theta^T(N)$ will be denoted by θ^T if no confusion can occur.

A **payoff vector** of a game (N,b,v) is a vector $x = (x_{i,k_i})_{(i,k_i)\in K^N}$, where x_{i,k_i} is the income of agent i at level k_i. For convenience, we define $x_{i,0} = 0$ for all $i \in N$. For $\alpha \in B^N$, we define $x(\alpha) = \sum_{i\in N}\sum_{k_i=1}^{\alpha_i} x_{i,k_i}$. Then x is **efficient (EFF)** if $x(b) = v(b)$. x is **level increase rational (LIR)** if $x_{i,k_i} \geq v(k_i\theta^{\{i\}}) - v((k_i-1)\theta^{\{i\}})$ for all $(i,k_i) \in K^N$. x is **individually rational (IR)** if $x(k_i\theta^{\{i\}}) \geq v(k_i\theta^{\{i\}})$ for all $(i,k_i) \in K^N$. x is **coalitionally**

rational (CR) if $x(\alpha) \geq v(\alpha)$ for all $\alpha \in B^N$.

EFF asserts that all agents allocate all the utility completely. **IR** asserts that the sum of the payoffs $\sum_{t=1}^{k_i} x_{i,t}$ for the agent i is at least the worth that the agent i could gain when he takes level k_i to participate alone. **LIR** asserts that the payoff x_{i,k_i} is at least the variation in worth that the agent i could gain when he participates alone and regulates the activity form level $k_i - 1$ to level k_i. **CR** asserts that the sum of the payoffs $x(\alpha)$ for all agents is at least the worth that all agents can gain when each agent i participates at operational level α_i. We denote the collection of **feasible payoff vectors** of (N,b,v) as $E^*(N,b,v) = \{x \in \mathbb{R}^{K^N} \mid x(b) \leq v(b)\}$, whereas $E(N,b,v)$ is the collection of efficient vectors of (N,b,v). Also, we denote $I^*(N,b,v) = \{x \in E(N,b,v) \mid x \text{ is } LIR \text{ and } IR\}$.

A **solution** on Γ is a function σ which associates with each game (N,b,v) a subset $\sigma(N,b,v) \subseteq E^*(N,b,v)$. The following definition is from Nouweland et al. (1995).

Definition 1. *Let* $(N,b,v) \in \Gamma$. *The* **core** $C(N,b,v)$ *consists of all efficient vectors* x *of* (N,b,v) *which satisfy LIR and CR, i.e.,* $C(N,b,v) = \{ x \in I^*(N,b,v) \mid x \text{ is } CR\}$.

3. Reductions and the *Pseudo Core*

Consider a payoff vector chosen by a solution for some game and a subgroup of players. Davis and Maschler (1965) defined the max-reduction as that each coalition in the subgroup can cooperate with a coalition outside the subgroup. In order to do so, it has to guarantee that the members outside of the subgroup can get at least their "original" payoffs. Moulin (1985) defined the complement-reduction as that in which each coalition in the subgroup could attain payoffs to its members only if they are compatible with the initial payoffs to "all" the members outside of the subgroup. Hwang et al. (2013) extended the max-reduction and the complement-reduction to multi-choice TU games as follows. Let x be a payoff vector of (N,b,v) and $S \in 2^N$, $S \neq \varnothing$. We denote $x|_S = (x_{i,k_i})_{(i,k_i) \in K^S}$. Let $(N,b,v) \in \Gamma$, $S \in 2^N, S \neq \varnothing$ and x be a payoff vector.

• The **DM-reduced game with respect to** S **and** x is the game $(S, b_S, v_{S,x}^{DM})$, where

$$v_{S,x}^{DM}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v(b) - \sum_{i \in N \setminus S} \sum_{k_i=1}^{b_i} x_{i,k_i} & \text{if } \alpha = b_S, \\ \max_{\beta \in B^{N \setminus S}} \{ v(\alpha, \beta) - \sum_{i \in N \setminus S} \sum_{k_i=1}^{\beta_i} x_{i,k_i} \} & \text{otherwise.} \end{cases}$$

• The **M-reduced game with respect to** S **and** x is the game $(S, b_S, v_{S,x}^{M})$, where

$$v_{S,x}^{M}(\alpha)=\begin{cases}0 & if\ \alpha=0_S\ ,\\ v(\alpha,b_{N\setminus S})-\sum_{i\in N\setminus S}\sum_{k_i=1}^{b_i}x_{i,k_i} & otherwise.\end{cases}$$

Consistency asserts that if x is assigned by σ for a game (N,b,v), then the projection of x to S should be assigned by σ for the reduction with respect to S and x for all $S\subseteq N$. Therefore, the projection of x to S should be consistent with the expectancies of the agents of S as reflected by their reduction.

• **DM-consistency (DMCON)**: For all $(N,b,v)\in\Gamma$, for all $S\in 2^N, S\neq\varnothing$, and for all $x\in\sigma(N,b,v)$, $(S,b_S,v_{S,x}^{DM})\in\Gamma$ and $x|_S\in\sigma(S,b_S,v_{S,x}^{DM})$.

"M-reduction" instead of "DM-reduction", we propose **M-consistency (MCON)**.

Different from the LIR property, we propose the RLIR property as follows. Let $(N,b,v)\in\Gamma$. A payoff vector x of (N,b,v) is **revised level increase rational (RLIR)** if $x_{i,k_i}\geq v(\alpha_{-i},k_i)-v(\alpha_{-i},k_i-1)$ for all $(i,k_i)\in K^N$ and for all $\alpha\in B^N$. **RLIR** asserts that the payoff x_{i,k_i} is at least the variation in worth that the agent i could gain when he or she participates in any condition and regulates the activity form level k_i-1 to level k_i. As we mentioned above, **LIR** focus on the variation of the payoff of each agent when he regulates each level and participates alone. But **RLIR** focus on the variation of the payoff of each agent when he regulates each level in any condition. Moreover, x is an **imputation** of (N,b,v) if it is EFF, RLIR and IR, and the collection of imputations of (N,b,v) is denoted by $I(N,b,v)$. Based on the notion of RLIR, we define the pseudo core as follows.

Definition 2. *Let* $(N,b,v)\in\Gamma$. *The* **pseudo core** $C_P(N,b,v)$ *consists of all efficient vectors* x *of* (N,b,v) *which satisfy RLIR and CR, i.e.,*

$$C_P(N,b,v)=\{\ x\in E(N,b,v)\mid x\ is\ RLIR\ and\ CR\ \}.$$

By focusing on the agents and the operational levels simultaneously, we also propose an extended reduction as follows. Let $(N,b,v)\in\Gamma$, $x\in\mathbb{R}^{K^N}$, $S\in 2^N, S\neq\varnothing$ and $\gamma\in B_+^{N\setminus S}$. The **agent-action reduced game with respect to** S **and** x is the game $(S,b_S,v_{S,x,\gamma}^{AA})$, where

$$v_{S,x,\gamma}^{AA}(\alpha)=\begin{cases}0 & if\ \alpha=0_S,\\ v(b)-\sum_{i\in N\setminus S}\sum_{k_i=1}^{b_i}x_{i,k_i} & if\ \alpha=b_S,\\ v(\alpha,\gamma)-\sum_{i\in N\setminus S}\sum_{k_i=1}^{\gamma_i}x_{i,k_i} & otherwise.\end{cases}$$

In the agent-action reduction, it is assumed that the coalition S with operational level α cooperates with all the agents of $N\setminus S$ with operational level γ, paying off each of them at initial

payoff x_{i,k_i}, where $(i,k_i) \in K^{N\setminus S}$. "Agent-action reduction" instead of "DM-reduction", we introduce **AA-consistency (AACON)**.

Converse consistency asserts that if the projection of an individual rational payoff vector x to every $S \subseteq N$ is consistent with the expectancies of the agents of S as reflected by their reduction, then x itself should be advocated for entire game. Based on the agent-action reduction, related converse consistency is defined as follows. Let σ be a solution on Γ.

• **Converse AA-consistency (CAACON)**: For all $(N,b,v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in I(N,b,v)$, if for all $S \subset N$ with $0 < |S| < |N|$ and for all $\gamma \in B_+^{N\setminus S}$ such that $(S,b_S,v_{S,x,\gamma}^{AA}) \in \Gamma$ and $x|_S \in \sigma(S,b_S,v_{S,x,\gamma}^{AA})$, then $x \in \sigma(N,b,v)$.

Remark 1. *Hwang et al. (2013) provided an example to show that the core violates DMCON and MCON as follows. Define a game* (N,b,v) *by* $N = \{i, j\}$, $b = (2,1)$ *and* $v(2,1) = 6$, $v(1,1) = 4$, $v(2,0) = v(0,1) = 2$, $v(1,0) = 1$ *and* $v(0,0) = 0$. *Let* $x \in C_P(N,b,v)$ *with* $x_{i,1} = 2$, $x_{i,2} = 1$ *and* $x_{j,1} = 3$. *Consider the reduced game* $(\{i\},2,v_{\{i\},x}^{DM})$, *it is easy to derive that* $v_{\{i\},x}^{DM}(2) = v(2,1) - x_{j,1} = 3$, and $v_{\{i\},x}^{DM}(1) = v(1,1) - x_{j,1} = v(1,0) = 1$. *So,* $v_{\{i\},x}^{DM}(2) - v_{\{i\},x}^{DM}(1)$ $= 3 - 1 = 2 > 1 = x_{i,2}$. *That is,* $(x_{i,1}, x_{i,2})$ *is not LIR in the reduced game* $(\{a\},2,v_{\{a\},x}^{DM})$. *The core violates DMCON. Similarly, the core violates DMCON. Similar to above example, it is easy to show that the pseudo core violates DMCON and MCON. By assuming* $\gamma = b_{N\setminus S}$ *in* $(S,b_S,v_{S,x,\gamma}^{AA})$, *the core violates AACON.*

4. Axiomatization and Non-emptiness

In this section, we adopt AACON and CAACON to axiomatize the pseudo core. Further, a necessary and sufficient circumstance for the non-emptiness of the pseudo core is also proposed.

Lemma 1. *Let* $(N,b,v) \in \Gamma$, $S \in 2^N$, $S \neq \varnothing$, $\gamma \in B_+^{N\setminus\{S\}}$ *and* x *be a payoff vector of* (N,b,v).

(1) If x is EFF in (N,b,v), then $x|_S$ is EFF in $(S,b_S,v_{S,x,\gamma}^{AA})$.

(2) If x is RLIR in (N,b,v), then $x|_S$ is RLIR in $(S,b_S,v_{S,x,\gamma}^{AA})$.

(3) If x is CR in (N,b,v), then $x|_S$ is CR in $(S,b_S,v_{S,x,\gamma}^{AA})$.

Proof. Let $(N,b,v) \in \Gamma$, $S \in 2^N$, $S \neq \varnothing$, $\gamma \in B_+^{N\setminus\{S\}}$ and x be a payoff vector of (N,b,v).

To verify (1), let $x \in E(N,b,v)$. By definition of $v_{S,x,\gamma}^{AA}$ and $x \in E(N,b,v)$,

$$v_{S,x,\gamma}^{AA}(b_S) = v(b) - \sum_{i \in N\setminus S} \sum_{k_i=1}^{b_i} x_{i,k_i} = \sum_{i \in S} \sum_{k_i=1}^{b_i} x_{i,k_i} = x|_S\,(b_S).$$

That is, $x|_S \in E(S,b_S,v_{S,x,\gamma}^{AA})$.

To verify (2), let x be RLIR in (N,b,v). For all $\alpha \in B^S$ and for all $(i,k_i) \in K^S$,

$$\begin{aligned} & v^{AA}_{S,x,\gamma}(\alpha_{-i},k_i) - v^{AA}_{S,x,\gamma}(\alpha_{-i},k_i-1) \\ = \quad & v((\alpha_{-i},k_i),\gamma) - \sum_{i\in N\setminus S}\sum_{k_i=1}^{\gamma_i} x_{i,k_i} - v((\alpha_{-i},k_i-1),\gamma) - \sum_{i\in N\setminus S}\sum_{k_i=1}^{\gamma_i} x_{i,k_i} \\ = \quad & v((\alpha_{-i},k_i),\gamma) - v((\alpha_{-i},k_i-1),\gamma). \end{aligned}$$

Since x is RLIR, $x_{i,k_i} \geq v((\alpha_{-i},k_i),\gamma) - v((\alpha_{-i},k_i-1),\gamma)$. That is, $x_{i,k_i} \geq v^{AA}_{S,x,\gamma}(\alpha_{-i},k_i) - v^{AA}_{S,x,\gamma}(\alpha_{-i},k_i-1)$ for all $\alpha \in B^{N\setminus S}$. Hence, $x|_S$ is RLIR in $(S,b_S,v^{AA}_{S,x,\gamma})$.

To verify (3), let x be CR in (N,b,v). For all $\alpha \in B^S$,

$$v^{AA}_{S,x,\gamma}(\alpha) - \sum_{i\in A(\alpha)}\sum_{k_i=1}^{\alpha_i} x_{i,k_i} = v(\alpha,\gamma) - \sum_{i\in N\setminus S}\sum_{k_i=1}^{\gamma_i} x_{i,k_i} - \sum_{i\in A(\alpha)}\sum_{k_i=1}^{\alpha_i} x_{i,k_i}.$$

Since x is CR, $v(\alpha,\gamma) - \sum_{i\in N\setminus S}\sum_{k_i=1}^{\gamma_i} x_{i,k_i} - \sum_{i\in A(\alpha)}\sum_{k_i=1}^{\alpha_i} x_{i,k_i} \leq 0$. That is, $x|_S(\alpha) \geq v^{AA}_{S,x,\gamma}(\alpha)$ for all $\alpha \in B^{N\setminus S}$. Hence, $x|_S$ is CR in $(S,b_S,v^{AA}_{S,x,\gamma})$.

Lemma 2. *On Γ, the pseudo core C_P satisfies AACON.*

Proof. This proof could be completed similarly as in Lemma 1.

Lemma 3. *On Γ, the pseudo core C_P satisfies CAACON.*

Proof. Let $(N,b,v) \in \Gamma$ with $|N| \geq 2$ and let $x \in I(N,b,v)$. Assume that $(S,b_S,v^{AA}_{S,x,\gamma}) \in \Gamma$ and $x|_S \in C_P(S,b_S,v^{AA}_{S,x,\gamma})$ for all $S \subset N$, $0 < |S| < |N|$ and for all $\gamma \in B^{N\setminus S}_+$. We will show that $x \in C_P(N,b,v)$. Since $x \in I(N,b,v)$, it remains to show that $x(\alpha) \geq v(\alpha)$ for all $\alpha \in B^N$. Two cases can be distinguished:

Case (1): $|N| = 2$:

Suppose that $N = \{i, j\}$. Let $\alpha = (\alpha_i,\alpha_j) \in B^N$.

• Consider $\alpha = (\alpha_i,\alpha_j) \in B^N$ with $\alpha_t = 0$ or $\alpha_t = b_t$ where $t \in \{i, j\}$. Since $x \in I(N,b,v)$, $x(\alpha) \geq v(\alpha)$.

• Consider $\alpha = (\alpha_i,\alpha_j) \in B^N$ with $\alpha_i \neq b_i$. Then

$$\sum_{k_i=1}^{\alpha_i} x_{i,k_i} \geq v^{AA}_{\{i\},x,\alpha_j}(\alpha_i) = v(\alpha_i,\alpha_j) - \sum_{k_j=1}^{\alpha_j} x_{j,k_j}$$

Hence, $x(\alpha) \geq v(\alpha)$.

• Consider $\alpha = (\alpha_i,\alpha_j) \in B^N$ with $\alpha_j \neq b_j$. The proof is similar to the previous arguments by considering the reduction $(\{j\},b_j,v^{AA}_{\{j\},x,\alpha_i})$.

Case (2): $|N|>2$:

Let $\alpha \in B^N \setminus \{0_N\}$. If $\alpha = b$, then the proof could be completed by $x \in E(N,b,v)$. Assume that $\alpha \neq b$.

• Assume that there exists $i \in N$ such that $0 < \alpha_i < b_i$. Consider the reduction $(\{i\}, b_i, v^{AA}_{\{i\},x,\alpha_{N\setminus\{i\}}})$. Then

$$\begin{aligned}\sum_{k_i=1}^{\alpha_i} x_{i,k_i} &\geq v^{AA}_{\{i\},x,\alpha_{N\setminus\{i\}}}(\alpha_i) \quad (By\ x|_{\{i\}} \in C_P(\{i\}, b_i, v^{AA}_{\{i\},x,\alpha_{N\setminus\{i\}}})) \\ &= v(\alpha_i, \alpha_{N\setminus\{i\}}) - \sum_{j\in N\setminus\{i\}} \sum_{k_j=1}^{\alpha_j} x_{j,k_j} \\ &\geq v(\alpha) - \sum_{j\in N\setminus\{i\}} \sum_{k_j=1}^{\alpha_j} x_{j,k_j}.\end{aligned}$$

Hence, $x(\alpha) \geq v(\alpha)$.

• Assume that $\alpha_i = 0$ or $\alpha_i = b_i$ for all $i \in N$. Assume that $\alpha_i = b_i$ and $\alpha_j = 0$ where $i, j \in N$. Consider the reduction $(S, b_S, v^{AA}_{S,x,\alpha_{N\setminus S}})$, where $S = \{i, j\}$. Then

$$\begin{aligned}\sum_{k_i=1}^{\alpha_i} x_{i,k_i} &\geq v^{AA}_{S,x,\alpha_{N\setminus S}}(\alpha_i, 0) \quad (By\ x|_S \in C_P(S, b_S, v^{AA}_{S,x,\alpha_{N\setminus S}})) \\ &= v(\alpha) - \sum_{t\in N\setminus S} \sum_{k_t=1}^{\alpha_t} x_{t,k_t} \\ &= v(\alpha) - \sum_{t\in N\setminus\{i\}} \sum_{k_t=1}^{\alpha_t} x_{t,k_t} \quad (Since\ \alpha_j = 0).\end{aligned}$$

Hence, $x(\alpha) \geq v(\alpha)$.

In order to axiomatize the pseudo core, some more axioms are needed.

Let σ be a solution on Γ.

• **Efficiency (EFFY)**: $\sigma(N,b,v) \subseteq E(N,b,v)$ for all $(N,b,v) \in \Gamma$.

• **Strong individual rationality (SIRY)**: $\sigma(N,b,v) \subseteq I(N,b,v)$ for all $(N,b,v) \in \Gamma$.

• **One-person rationality (OPRY)**: $\sigma(N,b,v) = I(N,b,v)$ for all $(N,b,v) \in \Gamma$ with $|N|=1$.

In general, OPRY does not always coincide with SIRY. But a solution σ satisfies SIRY if it satisfies OPRY for all one-person games. The OPRY of a solution could be adopted to guarantee the non-emptiness of a solution for all one-person games.

Lemma 4. *If a solution σ satisfies OPRY and AACON, then it satisfies EFFY.*

Proof. Let $(N,b,v) \in \Gamma$, $i \in N$ and $x \in \sigma(N,b,v)$. Consider the reduction

$(\{i\}, b_i, v^{AA}_{\{i\},x,b_{N\setminus\{i\}}})$. Obviously, $v^{AA}_{\{i\},x,b_{N\setminus\{i\}}}(b_i) = v(b) - \sum_{t\in N\setminus\{i\}}\sum_{k_t=1}^{b_t} x_{t,k_t}$. By AACON of σ, $x|_{\{i\}} \in \sigma(\{i\}, b_i, v^{AA}_{\{i\},x,b_{N\setminus\{i\}}})$. By OPRY of σ, $\sum_{k_i=1}^{b_i} x_{i,k_i} \geq v^{AA}_{\{i\},x,b_{N\setminus\{i\}}}(b_i) = v(b) - \sum_{t\in N\setminus\{i\}}\sum_{k_t=1}^{b_t} x_{t,k_t}$. That is, $x(b) \geq v(b)$. Since σ is a solution, $\sigma(N,b,v) \subseteq E^*(N,b,v)$. That is, $x(b) \leq v(b)$. Thus, $x(b) = v(b)$.

Theorem 1. *A solution σ satisfies OPRY, SIRY, AACON and CAACON if and only if*

$$\sigma(N,b,v) = C_P(N,b,v) \text{ for all } (N,b,v) \in \Gamma.$$

Proof. By Lemmas 2 and 3, the pseudo core satisfies AACON and CAACON. Clearly, the pseudo core satisfies OPRY and SIRY.

To complete the uniqueness, assume that a solution σ satisfies OPRY, SIRY, AACON and CAACON. Clearly, σ satisfies EFFY by Lemma 4. Let $(N,b,v) \in \Gamma$. We will complete the proof by math induction on $|N|$. If $|N| = 1$, then $\sigma(N,b,v) = I(N,b,v) = C_P(N,b,v)$ by OPRY of σ and C_P. Assume that $\sigma(N,b,v) = C_P(N,b,v)$ if $|N| < k$, $k \geq 2$.

The case $|N| = k$:

First we show that $\sigma(N,b,v) \subseteq C_P(N,b,v)$. Let $x \in \sigma(N,b,v)$. Since σ satisfies SIRY and EFFY, $x \in I(N,b,v)$. By AACON of σ, $x|_S \in \sigma(S, b_S, v^{AA}_{S,x,\gamma})$ for all $S \subset N$ with $0 < |S| < |N|$ and for all $\gamma \in B_+^{N\setminus S}$. By the induction hypothesis, $x|_S \in \sigma(S, b_S, v^{AA}_{S,x,\gamma}) = C_P(S, b_S, v^{AA}_{S,x,\gamma})$ for all $S \subset N$ with $0 < |S| < |N|$ and for all $\gamma \in B_+^{N\setminus S}$. By CAACON of C_P, $x \in C_P(N,b,v)$. The opposite inclusion could be shown alike by exchanging the positions of σ and C_P. Hence, $\sigma(N,b,v) = C_P(N,b,v)$.

The following examples present that each of the axioms adopted in Theorem 1 is logically independent of the others.

Example 1. *Let $\sigma(N,b,v) = \varnothing$ for all $(N,b,v) \in \Gamma$. Then σ satisfies SIRY, AACON and CAACON, but it violates OPRY.*

Example 2. *Define the solution σ to be that* $\sigma(N,b,v) = \begin{cases} I(N,b,v) & \text{, if } |N| = 1, \\ E(N,b,v) & \text{, otherwise.} \end{cases}$

Then σ satisfies OPRY, AACON and CAACON, but it violates SIRY.

Example 3. *Let $\sigma(N,b,v) = I(N,b,v)$ for all $(N,b,v) \in \Gamma$. Then σ satisfies OPRY, SIRY and CAACON, but it violates AACON.*

Example 4. *Define the solution σ to be that* $\sigma(N,b,v) = \begin{cases} I(N,b,v) & \text{, if } |N| = 1, \\ \varnothing & \text{, otherwise.} \end{cases}$

Then σ satisfies OPRY, SIRY and AACON, but it violates CAACON.

Inspired by Bondareva (1963) and Shapley (1967), we offer a necessary and sufficient circumstance for the non-emptiness of the pseudo core.

Definition 3. *A game* (N,b,v) *is said to be* **balanced** *if for all maps* $\lambda : B^N \to \mathrm{R}_+$ *and for all maps* $\eta : K^N \to \mathrm{R}_+$ *such that for all* $(i,k_i) \in K^N$,

$$\eta((i,k_i)) + \sum_{\substack{\alpha \in B^N \setminus \{\theta^{\{i\}}\} \\ \alpha_i \le k_i}} \lambda(\alpha) = 1 \qquad (1)$$

it holds that

$$\sum_{(i,k_i)\in K^N} [\eta((i,k_i)) \cdot \max_{\alpha \in B^N} \{v(\alpha_{-i},k_i) - v(\alpha_{-i},k_i - 1)\}] + \sum_{\substack{\alpha \in B^N \\ \mathrm{P}\alpha\mathrm{P} \ne 1}} \lambda(\alpha) \cdot v(\alpha) \le v(b).$$

Theorem 2. $C_P(N,b,v) \ne \varnothing$ *if and only if* (N,b,v) *is balanced.*

Proof. $C_P(N,b,v) \ne \varnothing$ if and only if there exists $x \in \mathrm{R}^{K^N}$ such that

$$x(b) = v(b) \qquad (2)$$

and for all $\alpha \in B^N$, for all $(i,k_i) \in K^N$,

$$\begin{aligned} &x(\alpha) \ge v(\alpha), \\ &x_{i,k_i} \ge v(\alpha_{-i},k_i) - v(\alpha_{-i},k_i - 1) \end{aligned} \qquad (3)$$

Inequality (3) coincides with that for all $\alpha \in B^N$ and for all $(i,k_i) \in K^N$,

$$\begin{aligned} &x(\alpha) \ge v(\alpha), \\ &x_{i,k_i} \ge \max_{\beta \in B^N} \{v(\beta_{-i},k_i) - v(\beta_{-i},k_i - 1)\} \end{aligned} \qquad (4)$$

Let $\Delta = \{x \mid x \text{ satisfies equation } (4)\}$. Then, there exists $x \in \mathrm{R}^{K^N}$ satisfying equation (2) and inequality (4) if and only if there exists $x^* \in \Delta$ such that

$$v(b) = x^*(b) = \min\{x(b) \mid x \in \Delta\} \qquad (5)$$

Let $\Lambda = \{(\lambda,\eta) \mid \lambda : B^N \to \mathrm{R}_+, \eta : K^N \to \mathrm{R}_+ \text{ and } \lambda,\eta \text{ satisfy } (1)\}$. Based on the duality results of linear programming theory, equation (5) coincides with

$$v(b) = \max\left\{ \sum_{(i,k_i)\in K^N} [\eta((i,k_i)) \cdot \max_{\alpha \in B^N} \{v(\alpha_{-i},k_i) - v(\alpha_{-i},k_i - 1)\}] + \sum_{\alpha \in B^N, \mathrm{P}\alpha\mathrm{P} \ne 1} \lambda(\alpha) \cdot v(\alpha) \right\}.$$

Thus, (N,b,v) is balanced.

5. The Dominance Core and Some Coincidences

In this section, we propose the *dominance core*, and investigate some coincidences among the pseudo core and the dominance core.

Let $(N,b,v)\in\Gamma$, $x,y\in I(N,b,v)$ and $\alpha\in B^N$. We say that y **dominates** x **through** α, denoted by $y\ dom_\alpha\ x$, if $y(\alpha)\le v(\alpha)$, $y_{i,k_i}>x_{i,k_i}$ for all $i\in N$ and for all $k_i\le\alpha_i$. We say that x **dominates** y if there exists $\alpha\in B^N$ such that $x\ dom_\alpha\ y$.

Definition 4. *The* **dominance core** $C_D(N,b,v)$ *of* $(N,b,v)\in\Gamma$ *consists of all* $x\in I(N,b,v)$ *for which there exists no* $y\in I(N,b,v)$ *such that* y *dominates* x.

Remark 2. *Let* $(N,b,v)\in\Gamma$, $\alpha\in B^N$ *and* $x,y\in I(N,b,v)$. *If* $y\ \ dom_\alpha\ \ x$, *then by* $x,y\in I(N,b,v)$, $|\{i\in N\,|\,\alpha_i\ne 0\}|>1$ *and* $\alpha\ne b$.

Lemma 5. *For all* $(N,b,v)\in\Gamma$, $C_P(N,b,v)\subseteq C_D(N,b,v)$.

Proof. Assume that there exists $(N,b,v)\in\Gamma$ and $x\in C_P(N,b,v)$ such that $x\notin C_D(N,b,v)$. So we have that there exists $y\in I(N,b,v)$ and $\alpha\in B^N$ such that $y\ \ dom_\alpha\ \ x$. Therefore,

$$v(\alpha)\le x(\alpha)<y(\alpha)\le v(\alpha),$$

which exactly presents a contradiction. Thus, $x\in C_D(N,b,v)$.

A game (N,b,v) is said to be **normalized** if for all $(i,k_i)\in K^N$, $v(k_i\theta^{\{i\}})=0$. The **normalization** of (N,b,v), denoted by $(N,b,\breve{v})$, is defined as follows. For all $\alpha\in B^N$,

$$\breve{v}(\alpha)=v(\alpha)-\sum_{i\in N}v(\alpha_i\theta^{\{i\}}).$$

Remark 3. *It is easy to check that if* $(N,b,v)\in\Gamma$ *be normalized and* $x\in I(N,b,v)$, *then* $x(k_i\theta^{\{i\}})\ge 0$ *for all* $(i,k_i)\in K^N$.

Lemma 6. *Let* $(N,b,v)\in\Gamma$, $(N,b,\breve{v})$ *be the normalization of* (N,b,v) *and* x *be a payoff vector of* (N,b,v). *Define a payoff vector* y *to be that*

$y_{i,k_i}=x_{i,k_i}-[v(k_i\theta^{\{i\}})-v((k_i-1)\theta^{\{i\}})]$ *for all* $(i,k_i)\in K^N$. *Then*

1. $x\in I(N,b,v)$ if and only if $y\in I(N,b,\breve{v})$
2. $x\in C_P(N,b,v)$ if and only if $y\in C_P(N,b,\breve{v})$
3. $x\in C_D(N,b,v)$ if and only if $y\in C_D(N,b,\breve{v})$

Proof. It is easy to complete these results by applying the definitions of the imputation set, the pseudo core, the dominance core and the normalization. Hence, we omit it.

Theorem 3. *Let* $(N,b,v)\in\Gamma$ *with* $C_D(N,b,v)\ne\varnothing$. *Then* $C_P(N,b,v)=C_D(N,b,v)$ *if and only if the normalization* $(N,b,\breve{v})$ *of* (N,b,v) *satisfies* $\breve{v}(\alpha)\le\breve{v}(b)$ *for all* $\alpha\in B^N$.

Proof. By Lemma 6, it suffices to complete the proof for normalized games. Assume that $(N,b,v)\in\Gamma$ is normalized.

Assume that $C_P(N,b,v)=C_D(N,b,v)$ and let $x\in C_P(N,b,v)$. By Remark 3 and $x\in C_P(N,b,v)$, $x_{i,k_i}\geq 0$ and

$$v(\alpha)\leq x(\alpha)\leq \sum_{i\in N}\sum_{k_i=1}^{\alpha_i} x_{i,k_i}+\sum_{i\in N}\sum_{k_i=\alpha_i+1}^{b_i} x_{i,k_i}=x(b)=v(b)$$

for all $(i,k_i)\in K^N$ and for all $\alpha\in B^N$.

Now assume that $v(b)\geq v(\alpha)$ for all $\alpha\in B^N$. By Lemma 5, it remains to complete that $x\notin C_D(N,b,v)$ for all $x\in I(N,b,v)\setminus C_P(N,b,v)$. Let $x\in I(N,b,v)\setminus C_P(N,b,v)$. Since $x\notin C_P(N,b,v)$, there exists $\alpha\in B^N$ such that $v(\alpha)>x(\alpha)$. Define a payoff vector y to be that

$$y_{i,k_i}=\begin{cases} x_{i,k_i}+\dfrac{v(\alpha)-x(\alpha)}{\sum_{k\in N}\alpha_k} & if\ i\in N\ and\ 1\leq k_i\leq\alpha_i \\ \dfrac{v(b)-v(\alpha)}{\sum_{k\in N}(b_k-\alpha_k)} & if\ i\in N\ and\ \alpha_i+1\leq k_i\leq b_i. \end{cases}$$

Clearly, $y(b)=v(b)$. Since $v(\alpha)>x(\alpha)$, $v(b)\geq v(\alpha)$ and $x_{i,k_i}\geq 0$ for all $(i,k_i)\in K^N$, we have that $y_{i,k_i}\geq 0$ for all $(i,k_i)\in K^N$. Thus, for all $(i,k_i)\in K^N$, $y_{i,k_i}\geq 0=$ $v(k_i\theta^{\{i\}})-v((k_i-1)\theta^{\{i\}})$ and $y(k_i\theta^{\{i\}})\geq 0=v(k_i\theta^{\{i\}})$. Hence, y is RLIR and IR in (N,b,v). Since y is EFF, RLIR and IR in (N,b,v), $y\in I(N,b,v)$.

By definition of y, $y_{i,k_i}>x_{i,k_i}$ for all $i\in N$ and for all $1\leq k_i\leq\alpha_i$. Hence, $y(\alpha_i\theta^{\{i\}})\geq x(\alpha_i\theta^{\{i\}})$ for all $i\in N$. Clearly,

$$y(\alpha)=x(\alpha)+\sum_{i\in N}\sum_{k_i=1}^{\alpha_i}\frac{v(\alpha)-x(\alpha)}{\sum_{k\in N}\alpha_k}=v(\alpha).$$

Since $y\in I(N,b,v)$, $y(\alpha_i\theta^{\{i\}})\geq x(\alpha_i\theta^{\{i\}})$ for all $i\in N$ and $y(\alpha)=v(\alpha)$, we have that y dom_α x. Hence, $x\notin C_D(N,b,v)$.

Theorem 4. *For all balanced game* (N,b,v), $C_P(N,b,v)=C_D(N,b,v)$.

Proof. By Lemma 6, it suffices to complete the proof for normalized games. Assume that (N,b,v) is normalized and balanced. By the proof of Theorem 3, $C_P(N,b,v)\neq\varnothing$ implies that for all $\alpha\in B^N$, $v(\alpha)\leq v(b)$. Since (N,b,v) is balanced, $C_P(N,b,v)\neq\varnothing$. Therefore, $C_D(N,b,v)\neq\varnothing$ by Lemma 5. Hence, $C_P(N,b,v)=C_D(N,b,v)$ by Theorem 3.

6. Conclusions

This paper gets three primary conclusions, as illustrated below one by one.

1. Nouweland et al. (1995) referred to other applications of multi-choice TU games, such as a large building project with a deadline and a penalty for every day if this deadline is overtime. The date of completion depends on the effort of how all of the people plunged into the project: the greater they exert themselves, the sooner the project will be completed. This situation gives rise to a multi-choice TU game. The worth of a coalition where each player works at a certain activity level is defined as the minus of the penalty which needs to be paid for giving the date of completion of the project when every player makes the relative effort.

An interesting application of the pseudo core could be modeled as follows. Let N={1,2,...,n} be a collection of all departments of a organization that could be equipped by various coalitions conjunctly. Let $B_i = \{0,1,\cdots,b_i\}$ be the operational levels collection of department i and let $v(\alpha)$ be the profit of offering the operational vector $\alpha = (\alpha_i)_{i\in N}$ in N conjunctly. The mapping v could be treated as an utility mapping which assigns to each α the worth that the departments can gain when each department i participates at the operational level α_i. Modeled in this notion, a management problem could be treated as a multi-choice TU games. An illustrative example is as follows.

Example 5. *On a synthetic investment plan, each investor* i *takes his strategy* t *to invest a sub-project of the investment plan, where* $t \in \{0,1,\cdots,b_i\}$. *The total funds from all investors are adopted to invest in some Build-Operate-Transfer (BOT) infrastructure plans. At the finishing of the investment plan, the government and the investors would share the beneficial bonus of the investment plan. If the investment plan would not make a profit this year, then each investor should shoulder the loss. By applying some explanations proposed in this paper, OPRY, SIRY, AACON and CAACON are fair and useful properties for allocation rules. Based on Theorem 1, each investor could decide that how much amount he should invest by applying the pseudo core.*

2. By focusing on the agents and the operational levels simultaneously, we proposed an extended reduction due to Moulin (1985). This appears the guess whether the other pre-existing reductions could also be presented in our configuration. Here we offer other extended reductions due to Davis and Maschler (1965), Serrano and Volij (1998) and Voorneveld and Nouweland (1998) as follows. Let $(N,b,v)\in\Gamma$, $x\in \mathrm{R}^{K^N}$, $S\in 2^N, S\neq\varnothing$ and $\gamma\in B_+^{N\setminus S}$.

- The **AA-DM-reduced game with respect to** S **and** x is the game $(S,b_S,v_{S,x,\gamma}^{AADM})$, where

$$v_{S,x,\gamma}^{AADM}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v(b) - \sum_{i\in N\setminus S}\sum_{k_i=1}^{b_i} x_{i,k_i} & \text{if } \alpha = b_S, \\ \max_{Q\subseteq N\setminus S}\{v(\alpha,\gamma_Q,0_{N\setminus S\setminus Q}) - \sum_{i\in Q}\sum_{k_i=1}^{\gamma_i} x_{i,k_i}\} & \text{otherwise} \end{cases}$$

• The **AA-SV-reduced game with respect to** S **and** x is the game $(S, b_S, v^{AASV}_{S,x,\gamma})$, where

$$v^{AASV}_{S,x,\gamma}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ \max\limits_{Q \subseteq N \setminus S} \{v(\alpha, \gamma_Q, 0_{N \setminus S \setminus Q}) - \sum\limits_{i \in Q} \sum\limits_{k_i=1}^{\gamma_i} x_{i,k_i}\} & \text{otherwise.} \end{cases}$$

• The **AA-VN-reduced game with respect to** S **and** x is the game $(S, b_S, v^{AAVN}_{S,x,\gamma})$, where

$$v^{AAVN}_{S,x,\gamma}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v(b) - \sum\limits_{i \in N \setminus S} \sum\limits_{k_i=1}^{b_i} x_{i,k_i} & \text{if } \alpha = b_S, \\ \max\limits_{\substack{Q \subseteq N \setminus S \\ Q \neq \varnothing}} \{v(\alpha, \gamma_Q, 0_{N \setminus S \setminus Q}) - \sum\limits_{i \in Q} \sum\limits_{k_i=1}^{\gamma_i} x_{i,k_i}\} & \text{otherwise.} \end{cases}$$

Similar to Remark 1, it is easy to check that the core and the pseudo core violate the consistency properties related to AA-DM-reduction, AA-SV-reduction and AA-VN-reduction.

3. In this paper, we build on the works of Nouweland et al. (1995) and Hwang et al. (2013). The techniques of several results in our paper are immediate analogues of those of Hwang et al. (2013). There are two major differences between their results and ours.

• By only considering the agents, Hwang et al. (2013) proposed two extended reductions on multi-choice TU games. In this paper, several extended reductions are proposed by focusing on the agents and the levels simultaneously.

• Different from the extended cores proposed by Nouweland et al. (1995) and Hwang et al. (2013), we propose two alternative extensions of the core on multi-choice TU games. Further, several coincidences among these two extended cores are also investigated.

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